Pebbling Graphs of Diameter Three and Four

Luke Postle*, Noah Streib†, and Carl Yerger‡

December 12, 2011

Abstract

Given a configuration of pebbles on the vertices of a connected graph G, a *pebbling move* is defined as the removal of two pebbles from some vertex and the placement of one of these on an adjacent vertex. The *pebbling number* of a graph G is the smallest integer k such that for each vertex v and each configuration of k pebbles on G there is a sequence of pebbling moves that places at least one pebble on v. We improve on a bound of Bukh by showing that the pebbling number of a graph of diameter three on n vertices is at most $\lfloor 3n/2 \rfloor + 2$, and this bound is best possible. Further, we obtain an asymptotic bound of $3n/2 + \Theta(1)$ for the pebbling number of graphs of diameter four. Finally, we prove an asymptotic bound for pebbling graphs of arbitrary diameter, namely that the pebbling number for a diameter d graph on n vertices is at most $(2^{\lceil \frac{d}{2} \rceil} - 1)n + C'(d)$, where $C'(d)$ is a constant depending upon d. This also improves another bound of Bukh.

1 Introduction

A recent development in graph theory, suggested by Lagarias and Saks (via a private communication to Chung), is called *pebbling*. Pebbling was first introduced into the literature by Chung who computed the pebbling number of Cartesian products of paths to give a combinatorial proof of the following number-theoretic statement of Kleitman and Lemke.

Theorem 1. *[2][8] Let $\mathbb{Z}_n$ be the cyclic group on n elements and let $|g|$ denote the order of a group element $g \in \mathbb{Z}_n$. For every sequence $g_1, g_2, \ldots, g_n$ of (not necessarily distinct) elements of $\mathbb{Z}_n$, there exists a zero-sum subsequence $(g_k)_{k \in K}$, such that $\sum_{k \in K} \frac{1}{|g_k|} \leq 1$. Here K is the set of indices of the elements in the subsequence.*

Chung developed the pebbling game to give a more natural proof of this theorem. Theorems of this type play an important role in this area of number theory as they generalize zero-sum theorems such as the Erdős-Ginzburg-Ziv [4] theorem. Over the last twenty years, pebbling has developed into its own subfield [6] [7], with over sixty papers.

Given a connected graph G, distribute k *pebbles* (indistinguishable markers) on its vertices in some *configuration* p. Specifically, a configuration on a graph G is a function

*School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, ljpostle@math.gatech.edu

Partially supported by NSF Graduate Research Fellowship

†School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, nstreib3@math.gatech.edu

‡Department of Mathematics, Davidson College, Davidson, NC 28036, math@carlyerger.com

Partially supported by NSF Graduate Research Fellowship while at Georgia Tech.

Keywords: pebbling, diameter, asymptotic

from $V(G)$ to $\mathbb{N} \cup \{0\}$. A *pebbling move* is defined as the removal of two pebbles from some vertex and the subsequent placement of one of those pebbles on an adjacent vertex.

A *rooted graph* is a pair (G, r) where G is a graph and $r \in V(G)$ is the *root vertex*. We say a pebbling configuration p is *potent* for a rooted graph (G, r) if there exists a pebbling configuration p' obtained by a sequence of pebbling moves from p such that r has at least one pebble in p'. We say a pebbling configuration p is *impotent* if there does not exist such a pebbling configuration. Define the *pebbling number* $\pi(G)$ to be the least integer k such that, for any vertex $v \in V(G)$ and any initial configuration p of k pebbles, p is potent for (G, v). Notice that a trivial lower bound for $\pi(G)$ is $|V(G)|$ — if one pebble is placed on each vertex of any subset of $|V(G)| - 1$ vertices, then we can define the root vertex to be the unpebbled vertex and we cannot make any pebbling moves to send a pebble to that vertex.

The pebbling number depends crucially on the diameter of the graph. For instance, the pebbling number of a path on n vertices is 2^{n-1}. Define $f(n, d)$ to be the maximum pebbling number of a diameter d graph on n vertices. Pachter, Snevily and Voxman in [9] proved that $f(n, 2) = n + 1$. Clarke, Hochberg and Hurlbert in [3] classified graphs of diameter two whose pebbling number is $n + 1$. More recently, Bukh [1] proved that $f(n, 3) = 3n/2 + O(1)$. We prove an exact bound for $f(n, 3)$, namely that $f(n, 3) = \lfloor 3n/2 \rfloor + 2$. We also give short proofs using our techniques of known results concerning graphs of diameter two.

Furthermore, we obtain two new asymptotic bounds, one for arbitrary diameter and one for graphs of diameter four. First, we prove an asymptotic bound for pebbling graphs of arbitrary diameter, namely $f(n, d) \leq (2^{\lceil \frac{d}{2} \rceil} - 1)n + 2^{4d} + 1$. This improves the bound of Bukh, who proved that $f(n, d) \leq (2^{\lceil \frac{d}{2} \rceil} - 1)n + D(n, d)$, where D is a function of order $\mathcal{O}(\sqrt{n})$ that depends on n and d. For graphs of diameter four, Bukh's general bound gives $f(n, 4) \leq 3n + D(n, 4)$, while our general bound gives $f(n, 4) \leq 3n + 2^{16} + 1$. However, using techniques from our arbitrary-diameter result, we show that $f(n, 4) = 3n/2 + \Theta(1)$.

2 Preliminaries and Terminology

Let (G, r) be a rooted graph with configuration p; that is $p : V(G) \mapsto \mathbb{N} \cup \{0\}$, where $p(v)$ is the number of pebbles on v in p. For vertices u and v, let $d(u, v)$ be the distance from u to v; that is, the length of the shortest path in G from u to v. When v is the root vertex r we simply write $d(u)$ instead of $d(u, r)$.

Define the *excess* of a set of vertices S, denoted $X(S)$, by

$$X(S) := \sum_{v \in S} (p(v) - 1.5).$$

With this definition in mind, we say that a set S of k vertices is *p-suboptimal* if $X(S)$ is at most zero, and a set with positive excess is *p-superoptimal*. When p is clear from context, we will drop it from the notation.

A vertex w is a *parent* of an adjacent vertex v in rooted graph (G, r) when $d(w, r) = d(v, r) - 1$. Likewise, a vertex is a *child* of its parent. Given a vertex w, the set of *descendants* of w is the set of vertices v which can be put in a list $w = w_1, w_2, \ldots, w_k = v$, where $k > 1$ and w_{i-1} is a parent of w_i for all $i \geq 2$. Similarly, given vertex v the set of *ancestors* of v is the set of vertices w which can be put in a list $w = w_1, w_2, \ldots, w_k = v$, where $k > 1$ and v_{i-1} is a parent of v_i for all $i \geq 2$.

Let F be a subset of the edges of G and p a pebbling configuration on G. We say that a pebbling configuration p' is *F-reachable* if p' can be obtained from p by a sequence of pebbling moves which send pebbles only along the edges in F. We say a configuration is *reachable* if it is $E(G)$-reachable.

Let T be a breadth-first search (BFS) tree of a rooted graph (G, r) with pebbling configuration p. We say that the pair (B, p) is a *branch* of (G, r) when B is a subtree of T and p is restricted to the vertices of B. When p is understood we will simply say that B is a branch. The *base* of a branch is the unique vertex in the branch whose distance to r is minimal. Suppose $S \subseteq V(G)$. Define the pebbling configuration p_S as follows: $p_S(v) = p(v)$ if $v \in S$ and $p_S(v) = 0$ if $v \notin S$. For a configuration p, we define $p^*(v)$ to be the maximum of $p'(v)$ over all configurations p' that are reachable from p. To simplify notation when B is a branch, let $p^*_B(v) = p^*_{V(B)}(v)$; that is, $p^*_B(v)$ denotes the maximum number of pebbles a branch B can send to a vertex v.

The *pebbling capacity* of a branch (B, p) with base w is the maximum of $\lfloor p^*_B(w)/2 \rfloor$ taken over all configurations p' that are $E(B)$-reachable from p. A branch B, with base vertex w, is *irreducible* if for all vertices $v \in B$, where $v \neq w$, the branch induced by v and its descendants in B has nonzero capacity. The idea is that a larger branch could be reduced to components involving these irreducible branches.

Remark 1. For a vertex v in an irreducible branch, $p(v) + (\#$ of children of $v)$ is at most $2^{d(v)} - 1$.

The *depth* of branch B is the number of vertices in the longest path contained entirely in B between the base and a vertex in B.

A vertex v in a branch (B, p) of a rooted graph (G, r) is *k-heavy* if $p^*_B(v) \geq 2^k$. If G is a graph of diameter d, we will say that a $\lceil d/2 \rceil$-heavy vertex is *heavy*. Let $H(B)$ denote the set of heavy vertices of B.

Let (B, p) be an irreducible branch with pebbling capacity zero. For a vertex $v \in B$ we define $\mu_B(v)$, the *deficiency* of v in B, as the maximum of $p'(v) - p(v)$ over all configurations p' such that $p'(x) = p(x)$ for all $x \in V(B) \setminus \{v\}$, and (B, p') has pebbling capacity zero. That is, the deficiency of v is the maximum number of pebbles that can be added to v such that the pebbling capacity of the branch is still zero. Further, define $\mu(B)$ as the maximum of $\mu(v)$ over all $v \in B$.

Proposition 1. *Let (G, r) be a rooted graph with impotent pebbling configuration p. Let τ be a breadth-first search tree of G, rooted at r. Then we can partition the vertices of $G \setminus \{r\}$ using disjoint irreducible branches of τ with zero pebbling capacity. Further, the constructed partition of branches of τ is unique.*

Proof. Let Z be the set of vertices $z \in V(G) \setminus \{r\}$ such that the branch induced by z and its descendants in τ has zero pebbling capacity. For each $z \in Z$, let B_z be the maximal subtree of τ with base z that does not contain another vertex in Z. Note that B_z is a branch. Let $\mathcal{B}$ be the set of all such branches. We claim that $\mathcal{B}$ is our desired partition.

To this end, notice that the branches in $\mathcal{B}$ cover the vertices of $G \setminus \{r\}$, as each neighbor of r is in Z by the assumption that p is impotent, and all other vertices in $V(G)$ are descendants of neighbors of r. Furthermore, the fact that τ is a tree implies that no vertex is in more than one branch. So we indeed have a partition. By construction, each branch in $\mathcal{B}$ has zero pebbling capacity. The only vertex in a branch with zero pebbling capacity is the base, and hence each branch is irreducible. Lastly, the partition is unique since the set Z is well-defined. ☐

The unique partition constructed in Proposition 1 will be referred to as the *τ-marking* of (G, r). When τ is clear, we will refer to it simply as a *marking* of (G, r). Let B_w denote the branch containing non-base vertex w. Note that B_w is well-defined since the marking partitions $V(G) \setminus \{r\}$ with branches and the partition is unique. We say that a vertex is *solitary* if it is the only vertex in its branch. Note that in a graph of diameter d with configuration p, every vertex v such that $d(v, r) = d$ and $p(v) \leq 1$ is solitary.

Let L be the set of branches depicted in Figure 1. We shall adopt the following notation for branches: (1a) is 0-0-7, (1d) is 0-0-(3,5), (1f) is 0-0-(3,3,3), etc.

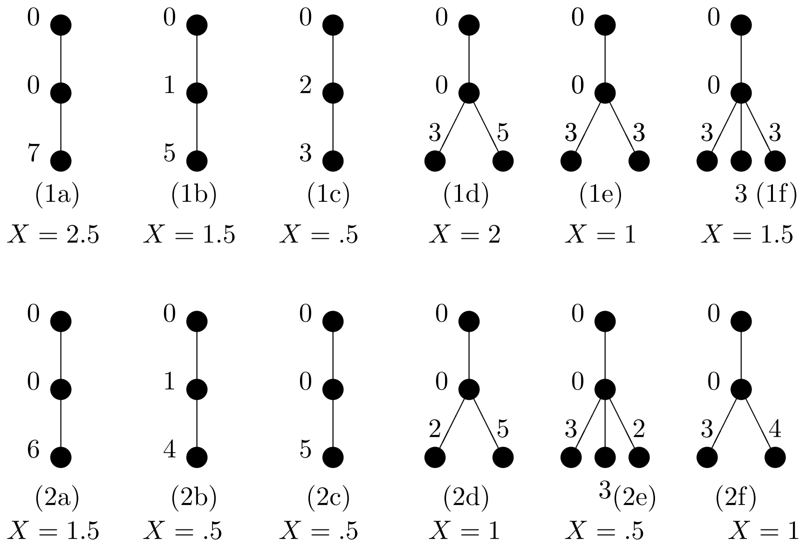

Figure 1: The set of branches defined by L. In this figure, the top vertex of every branch is the base. The top row describes the set of branches with zero deficiency everywhere, whereas the bottom row gives branches where there is nonzero deficiency in at least one vertex.

Lemma 1. *Let G be a graph with diameter at most three. Then (B, p) is an irreducible, superoptimal branch of G with pebbling capacity zero if and only if it is an element of L.*

Proof. By inspection, every branch in L is irreducible, superoptimal and has pebbling capacity zero. To prove the converse, let B be a superoptimal irreducible branch with pebbling capacity zero in a marking of (G, r) with base w. Observe that $p(w) \in \{0, 1\}$ as otherwise B has nonzero pebbling capacity. Since B is irreducible, if $p(w) = 1$, then w has no children. The branch $\{w\}$, where $p(w) = 1$, is suboptimal. So we conclude that $p(w) = 0$. If $|V(B)| \geq 2$, there is a unique child of w in B, say x. To see this, notice that, since B is irreducible, each vertex other than w in B must be able to obtain positive pebbling capacity by pebbling moves from its descendants in B. Thus, if there is more than one neighbor of w in B, then w could obtain at least two pebbles from these vertices, a contradiction. Similarly, $p(x) \leq 3$.

Suppose $p(x) = 3$. In this case, there can be no descendants of x in B, as each descendant must have positive pebbling capacity. However, the branch 0-3 is suboptimal, so we may assume $p(x) \leq 2$. Suppose $p(x) = 2$. In this case, by similar arguments, there can be at most one child of x, and it must contain two or three pebbles. This produces the branches 0-2-2 and 0-2-3, and only the latter is superoptimal. This is case (1c). Now suppose $p(x) = 1$. Notice that x can have at most two children in B. If there are two, then each has pebbling capacity one, producing configurations 0-1-(2,2), 0-1-(2,3), and 0-1-(3,3). Only the last configuration is superoptimal. This is case (1e). If x has one child, its pebbling capacity is at most two. This gives configurations 0-1-2, 0-1-3, 0-1-4, 0-1-5. Only the last two are superoptimal. This gives cases (2b) and (1b), respectively.

Finally, suppose that $p(x) = 0$. There are at most three children of x in B, as otherwise w can receive at least two pebbles. First, suppose there are three children. The pebbling capacity of each is exactly one. This gives the configurations 0-0-(2,2,2), 0-0-(2,2,3), 0-0-(2,3,3) and 0-0-(3,3,3). Only the last two are superoptimal. These are (2e) and (1f), respectively. Second, suppose there are two children. If each has pebbling capacity one, then we produce configurations 0-0-(2,2), 0-0-(2,3), 0-0-(3,3), which are all suboptimal. If one of the two children has pebbling capacity two, then we produce configurations 0-0-(2,4), 0-0-(2,5), 0-0-(3,4), and 0-0-(3,5). The last three configurations are superoptimal. They are (2d), (2f) and (1d), respectively. Finally, suppose there is only one child of x. This vertex has pebbling capacity at most three. Since x must have nonzero pebbling capacity, the feasible configurations are 0-0-4, 0-0-5, 0-0-6, 0-0-7. The last three are superoptimal. They produce configurations (2c), (2a) and (1a), respectively. □

Remark 2. The vertices with deficiency one in L are '6' in (2a), '4' in (2b), '2' in (2d), '4' in (2f), '2' in (2e), and the '0' distance two from r in (2c). The only vertex with deficiency two vertex is '5' in (2c).

Remark 3. Every branch in L has a heavy vertex.

3 Diameter Two Results

As a way to introduce the techniques used later in the paper, and to simplify known results, we will give a new proof of the fact that $f(n,2) = n+1$.

Remark 4. Notice that the only branches of pebbling capacity zero in a marking of a diameter two graph are the solitary branches 0, and 1, and the non-solitary branches 0-2, 0-3.

Theorem 2 (Pachter, Snevily, Voxman)**.** *Let G be a graph with n vertices and diameter two. Then $\pi(G) \leq n+1$. Furthermore, $f(n,2) = n+1$.*

Proof. First we show that $\pi(G) \leq n+1$ by showing that any impotent pebbling configuration on G has at most n pebbles. To this end, let $r \in V(G)$ and consider the rooted graph (G,r) with impotent pebbling configuration p. Let M be a marking of G. Since p is impotent, $p^*(v) \leq 3$ for all $v \in V(G)$. Let $T = \{v \in V(G) | p(v) = 3\}$ and $U = \{u \in V(G) | p(u) = 0 \text{ and } u \text{ solitary}\}$.

Let $t_1, t_2 \in T$. It must be that t_1 is not adjacent to t_2, otherwise t_1 could receive a pebble from t_2, a contradiction. Since G is diameter two, there must exist a vertex u adjacent to both t_1 and t_2. If $p(u) \neq 0$, t_1 could receive a pebble from t_2, a contradiction. Moreover, since u can receive a pebble from both t_1 and t_2, u is distance two from r. But this implies that u is solitary. Furthermore, there does not exist a vertex w with at least three neighbors, call them t_1, t_2, t_3 in T. Otherwise w could receive a pebble from both t_2 and t_3 and then send a pebble to t_1, a contradiction. Hence, it follows that $|U| \geq \binom{|T|}{2}$.

Note that $\sum_{x \in V(G)} p(x) = n - 1 + |T| - |U|$. This follows from Remark 4 as each vertex in T is part of a 0-3 branch that contributes three pebbles, one more than the number of vertices in its branch, to the sum. Similarly, each vertex in U contributes no pebbles to the sum, one less than the number of vertices in its branch. The other two types of branches, 1 and 0-2, contribute exactly the number of pebbles to the sum as the number of vertices in its branch.

However, $|T| - |U| - 1 \leq |T| - \binom{|T|}{2} - 1$ from above. Yet this equals $-(|T|-1)(|T|-2)/2$, which is always nonpositive and equal to zero only if $|T| = 1$ or 2. Hence, $\sum_x p(x) \leq n$.

To prove $f(n,2) = n+1$, we need to exhibit a graph of diameter two with pebbling number $|V(G)| + 1$. To this end, let G be the bipartite graph $K_{n-1,1}$. Let $\{v_1, \ldots, v_{n-1}\}$ be the vertices of the side of the partition of size $n-1$ and let v_n be the other vertex. Set $p(v_1) = 3$, $p(v_2) = \cdots = p(v_{n-2}) = 1$ and set $p(v_{n-1}) = p(n) = 0$. Let the root vertex be v_{n-1}. This configuration of n pebbles is clearly impotent, and hence $\pi(G) \geq n+1$. □

A trivial lower bound on the pebbling number of any graph is n, which is witnessed by the configuration that places one pebble on every vertex except for the root. Thus, Theorem 2 implies that the pebbling number of a graph of diameter two is either n or $n+1$. Next, in Theorem 3, we give a new proof of the characterization of those graphs of diameter two with pebbling number $n+1$, and hence those with n as well.

Theorem 3 (Clarke, Hochberg, Hurlbert)**.** *Let G be a graph of diameter two. Then $\pi(G) = n+1$ if and only if G has a cutvertex or G contains a six-cycle $C = v_1 \dots v_6$ such that for all i, j where $1 \le i < j \le 3$, $\{v_{2i}, v_{2j}\}$ is a 2-cut in G which separates C.*

Proof. Let $r \in V(G)$ be the root of G. If G has a cutvertex v that separates r from some $w \in V(G)$, we can define an impotent pebbling configuration p on (G, r) with n pebbles as follows: let $p(r) = 0$, $p(v) = 0$, $p(w) = 3$, and $p(x) = 1$ for all $x \neq r, v, w$. On the other hand, if G contains a six-cycle as described in the theorem, we can define an impotent pebbling configuration p on (G, v_5) with n pebbles as follows: let $p(v_1) = p(v_3) = 3$, $p(x) = 0$ for $x \in C \backslash \{v_1, v_3\}$, and let $p(x) = 1$ for all $x \notin C$.

So suppose that $\pi(G) = n+1$. Thus there exists an impotent pebbling configuration p with n pebbles on (G, r) for some $r \in V(G)$. Borrowing terminology from the proof of Theorem 2, either $|T| = 1$ and $|U| = 0$, or $|T| = 2$ and $|U| = 1$. First suppose $|T| = 1$ and $|U| = 0$, let $v \in T$ and let $w \neq v$ be in B_v. We claim that w separates v from r. Suppose not. Then there exists a path P from v to r in $G \setminus \{w\}$. Let z be the first vertex on P such that $p(z) \neq 1$. If $p(z) \ge 2$, we can send a pebble from z along P to v, a contradiction. If $p(z) = 0$, then we can send a pebble from v along P to z. Since z is not solitary, we know z is adjacent to r and z can receive a pebble from a child not in P. But then z can send a pebble to r, a contradiction. Hence, w is a cutvertex.

So we may assume $|T| = 2$ and $|U| = 1$. Let $a_1 = r$ and $T = \{a_2, a_3\}$. Let $b_3 \in B_{a_2} \backslash \{a_2\}$, and $b_2 \in B_{a_3} \backslash \{a_3\}$. Finally, let $b_1 \in U$. Using the proof of Theorem 2, we know $C = a_1 b_3 a_2 b_1 a_3 b_2$ is a six-cycle. We claim for all $\{i, j, k\} = \{1, 2, 3\}$ that $\{b_i, b_j\}$ separates $\{a_i, b_k, a_j\}$ from a_k. Suppose not. Then there exists a path P from $A = \{a_i, b_k, a_j\}$ to $B = \{a_k\}$ in $G \setminus \{b_i, b_j\}$. Notice that none of the vertices in C can receive a pebble from $V(G) \backslash C$ as then p would be potent. Moreover, either all the vertices in A or all the vertices in B can obtain two pebbles from pebbling moves inside that set. Call this set A_0 and the other B_0. Let z be the first vertex along P, starting from A_0, such that $p(z) \neq 1$. If $p(z) \ge 2$ we can send a pebble from z along P to A_0, a contradiction. If $p(z) = 0$ we can send a pebble from A_0 along P to z. Since z is not solitary, z is adjacent to r. But then z can send a pebble to r, a contradiction. □

4 Lower Bounds

To obtain tightness in our bounds for graphs of diameter three and four, we must construct graphs such that $f(n, 3) = \lfloor 3n/2 \rfloor + 2$ and $f(n, 4) = 3n/2 + \mathcal{O}(1)$. The following graphs achieve this for $f(n, 3)$ with p as shown. A variant of this graph appears explicitly in [1] and a similar graph is described in [5]. In Figure 2, the vertices in the box form a clique. Notice that we have placed $\lfloor 3n/2 \rfloor + 1$ pebbles on each of these configurations and yet we can not move a pebble to r. For $f(n, 4)$, increase the length of the leftmost branch of each graph in Figure 2 by one, and place fifteen pebbles on the bottom vertex of this branch while deleting the seven from its parent. This gives a diameter four graph with pebbling number $\lceil 3n/2 \rceil + 8$. This structure also gives the best known lower bound for $f(n, d)$, when we extend each of the branches to be length $\lceil d/2 \rceil$. This gives a bound of $f(n, d) \ge \frac{(2^{\lceil \frac{d}{2} \rceil} - 1)}{\lceil \frac{d}{2} \rceil} n + C(d)$, where $C(d)$ is a positive constant depending only upon d. As a result we have proved the following theorems:

Theorem 4. *The following inequality holds:*

$$f(n,3) \geq \lfloor 3n/2 \rfloor + 2.$$

Theorem 5. *For every positive integer d, there exists a constant C(d) such that*

$$f(n,d) \geq \frac{(2^{\lceil \frac{d}{2} \rceil} - 1)}{\lceil \frac{d}{2} \rceil} n + C(d).$$

In particular, if $d = 4$, $f(n,d) \geq \lceil 3n/2 \rceil + 8$.

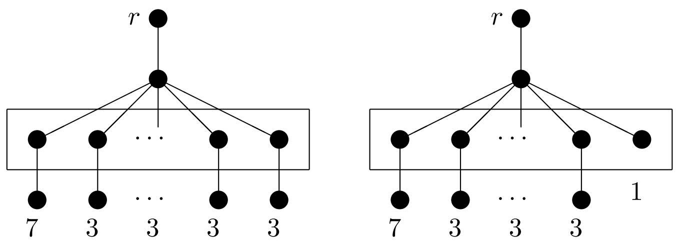

Figure 2: Lower bounds for n even and n odd.

5 Diameter Three Results

In this section, let (G, r) be a rooted graph of diameter three on n vertices. Let p be an impotent pebbling configuration on (G, r) and let M be a marking of (G, r) using p with respect to some breadth-first search tree τ. Since M partitions $V(G)\backslash\{r\}$ uniquely into branches, we will write $\mu(v)$ for the deficiency of v instead of $\mu_{B_v}(v)$. We seek to prove the following theorem.

Theorem 6. *Let G be a graph on n vertices with diameter three. Then* $\pi(G) \leq \lfloor 3n/2 \rfloor + 2$. *Furthermore,* $f(n,3) = \lfloor 3n/2 \rfloor + 2$.

It suffices to prove that $\pi(G) \leq \lfloor 3n/2 \rfloor + 2$, as Theorem 4 yields $f(n,3) \geq \lfloor 3n/2 \rfloor + 2$. To this end, we prove that $\sum_{x \in V(G)} p(x) \leq \lfloor 3n/2 \rfloor + 1$. We may assume there exists a superoptimal branch in M as otherwise the branches themselves would be a partition of $V(G)\backslash\{r\}$ into suboptimal sets, showing that G itself is suboptimal. Let B_0 be a superoptimal branch such that $\mu(B)$ is minimized over all branches in M. Let w_0 be a non-base vertex in B_0 such that $\mu(w_0)$ is minimized, and subject to that condition, $p(w_0)$ is maximized. Note that $\mu(w_0) = 0$ unless w_0 is the '0' distance two from r in (2c), in which case $\mu(w_0) = 1$.

Our goal is to show that most of the superoptimal branches have corresponding suboptimal branches whose negative excess balances out the positive excess of the superoptimal branches. As we will show, this will be sufficient in proving Theorem 6.

Definition. Let $B \neq B_0$ be a superoptimal branch in M. A vertex v is a *primary candidate of B* if

1. $d(v, w_0) \leq 1$,

2. $p^*_B(v) \geq 1$,

3. $v \notin B$.

Lemma 2. *Every superoptimal branch $B \neq B_0$ in M has a primary candidate.*

Proof. By Remark 3, B has a heavy vertex. Let $v \in H(B)$. Since v is heavy, $d(v) = 3$. Note that v is at least distance two from w_0, as otherwise v could send two pebbles to w_0, a contradiction as $\mu(w_0) \leq 1$. If v is distance two from w_0, then v can send a pebble to w_0. In this case, w_0 is a primary candidate of B. So suppose that $d(v, w_0) = 3$. Consider a shortest path between v and w_0 and let u be the vertex on the path adjacent to w_0. It is clear that $d(u, w_0) = 1$ and that v can send a pebble to u.

Now u is a primary candidate of B unless $u \in B$. So suppose $u \in B$. Note that u is not adjacent to r, as otherwise u could receive a pebble from v and from B_0 (as $p^*_{B_0}(w_0) \geq 2$) and then send a pebble to the root. Suppose $d(v) = 2$. Notice that in all branches of L, there is a unique vertex distance two from r. Hence, this must be vertex u in B. But then u is adjacent to v, and so $d(u, w_0) = 2$, which is considered in the previous paragraph. Thus it must be that $d(u) = 3$ as $d(u, v) = 2$. Hence, $p(u) \geq 2$ and B is not (2c). This implies that $\mu(w_0) = 0$ and yet u can send a pebble to w_0, a contradiction. □

Definition. Let $B \neq B_0$ be a superoptimal branch in M. A primary candidate v of B is a *primary goal* of B if

1. $p(v) = 0$,

2. v is solitary.

Notice that the conditions for a vertex v being a primary goal are equivalent to v being unable to obtain a pebble from its own branch B_v. That is, $p^*_{B_v}(v) = 0$.

Definition. Let $B \neq B_0$ be a superoptimal branch in M. A vertex v is a *secondary goal of B* if

1. $d(v, w_0) = 2$,

2. $p^*_B(v) \geq 2$,

3. $p(v) = 0$ and v is solitary.

The idea behind candidates and goals is that every superoptimal branch has a candidate vertex. A goal is something to aim for because we can group a goal with a superoptimal branch later in the proof so that their combined excess is nonpositive. We say that a vertex is *unshared* if it is a primary candidate of only one superoptimal branch $B \neq B_0$ or if it is the primary or secondary goal of only one superoptimal branch $B \neq B_0$. Notice that if a vertex v is a primary or secondary goal, then v is solitary. We want goals to be unshared because then it is possible to uniquely pair them with a superoptimal branch so that no two superoptimal branches share the same goal. Furthermore, $p(v) = 0$ and hence $X(B_v) = -1.5$.

Lemma 3. *If $B \neq B_0$ is a superoptimal branch in M such that $\mu(B) = 0$, then B has an unshared secondary goal.*

Proof. Note that $\mu(B_0) = 0$ since, by definition, $\mu(B_0) \leq \mu(B)$. Hence B and B_0 are isomorphic to one of (1a)-(1f). Also, $\mu(w_0) = 0$ and w_0 is heavy. Let v be a heavy vertex in B. Then v is distance three from w_0 as otherwise v could send a pebble to w_0. Thus there exists a vertex u adjacent to v such that $d(u, w_0) = 2$. Since u is adjacent to v,

$p^*_B(u) \geq 2$, which implies that $u \notin B_0$. Similarly since $d(u, w_0) = 2$, $p^*_{B_0}(u) \geq 1$, which implies that $u \notin B$.

Since w_0 can send a pebble to u and yet u cannot send a pebble to v, it must be that u cannot receive a pebble from its own branch. That is, $p^*_{B_u}(u) = 0$. This can only happen if u is solitary and $p(u) = 0$. Thus u is a secondary goal of B. Finally suppose u is a secondary goal for another superoptimal branch $B' \notin \{B_0, B\}$. Now B' could send two pebbles to u, and then u could send a pebble to v, a contradiction. Hence u is an unshared secondary goal of B. □

Lemma 4. *Suppose that $\mu(w_0) = 0$. If u is a primary candidate of a superoptimal branch $B \neq B_0$ in M, then u is unshared. Furthermore, if $u \notin B_0$, then u is an unshared primary goal of B.*

Proof. First suppose that u is a primary candidate of another superoptimal branch B'. Note that u could receive a pebble from both B and B' and thus send a pebble to w_0, a contradiction to the assumption that $\mu(w_0) = 0$. Hence u is unshared. Now suppose u could receive a pebble from its own branch. Since u is a primary candidate of B, u could also receive a pebble from B. Hence u could send a pebble to w_0, a contradiction. Thus $p^*_{B_u}(u) = 0$ and u is an unshared primary goal of B. □

Let $\mathcal{S}$ denote the set of superoptimal branches in M. Let $\mathcal{F}$ be the set of all primary and secondary goals of elements of $H \setminus B_0$.

Lemma 5. *If $\mu(w_0) = 1$, then $|\mathcal{F}| \geq |\mathcal{S}| - 5$.*

Proof. Since $\mu(w_0) = 1$, it must be that all superoptimal branches are isomorphic to (2c). Also, w_0 is the '0' distance two from r in (2c). By Lemma 2, every $B_i \in \mathcal{S} \setminus B_0$ has a primary candidate. Choose such a primary candidate of B_i and let u_i denote this vertex. Let U denote the set of all the u_i. Note that $u_i \notin B_j$ for all $i \neq j$. Otherwise, since u_i could receive two pebbles, one from each of B_i and B_j, u_i would have to have deficiency at least two. Hence u_i would have to be the '5' in B_j. But then u_i could send two pebbles to w_0, a contradiction.

Now we claim that there exists at most one vertex v such that $|\{u_j \in U | u_j = v\}| \geq 2$. Suppose not, then there exists a vertex v_1 that is a primary candidate of one pair of branches and another vertex v_2 that is a primary candidate of another distinct pair of branches. Then from their pairs, v_1 and v_2 can each receive two pebbles simultaneously. Each can then send a pebble to w_0, a contradiction.

Now suppose one such vertex v exists. We claim that every chosen primary candidate $u_i \neq v$ is a primary goal. Note that v can receive a pebble from each of the branches for which it is a candidate. Hence v can send a pebble to w_0. Thus none of the vertices in B_0 can receive any more pebbles as the root could then receive a pebble from B_0, a contradiction. This implies that $u_i \notin B_0$. It also implies that $p^*_{B_{u_i}}(u_i) = 0$. Otherwise, u_i could obtain a pebble from its own branch and receive a pebble from B_i. Then u_i could send a pebble to B_0, a contradiction to the statement above. Indeed, u_i is a primary goal. Since $|\{u_i \in U | u_i \neq v\}| \geq |\mathcal{S}| - 4$ (as otherwise v could send two pebbles to w_0) and every one of them is a primary goal, we find that $|\mathcal{F}| \geq |\mathcal{S}| - 4$.

So suppose that no such vertex v exists. We claim that there exist at most two vertices $u \in U \setminus V(B_0)$ such that u is not a primary goal. To see this, suppose there exist at least three vertices. Since these vertices are not primary goals, they can obtain a pebble from their own branch. Suppose that two of them, say u_1 and u_2 without loss of generality, are not in the same branch. Then u_1 and u_2 can each obtain two pebbles simultaneously; u_1 can receive one from B_{u_1} and one from B_1 while u_2 can receive one from B_{u_2} and one from B_2. But then each can send a pebble to w_0, a contradiction.

So suppose all three are in the same branch B'. We claim that none of these three vertices can be the base. Suppose that vertex u_3 was the base. Then it receives two pebbles as follows. Since w_0 receives two pebbles from B_0, a pebble is sent to u_3 from w_0. An additional pebble is sent to u_3 as it is the base of branch B'. Thus, none of the three vertices can be the base. However, since B' has diameter two, then at least two of the vertices, say u_1 and u_2 are on the same level. As a result, their induced subbranches are disjoint and have nonzero capacity. Thus they can each obtain two pebbles simultaneously as above and then each can send a pebble to w_0, a contradiction.

Note that the first '0' in B_0 has deficiency zero and hence cannot be a primary candidate. Thus at most four of the vertices in U are not primary goals, as there are at most two such vertices not in B_0 and at most two in B_0. Since each element in U is chosen by a unique element in $\mathcal{S} \setminus B_0$, we find that $|\mathcal{F}| \geq |H| - 5$. □

We are now prepared to prove Theorem 6.

Proof of Theorem 6. It is sufficient to prove that $X(\mathcal{F} \cup \bigcup_{B \in \mathcal{S}} V(B)) \leq +2.5$, as all suboptimal branches have nonpositive excess.

First suppose that $\mu(B_0) = 0$. In this case, B_0 is isomorphic to one of (1a)-(1f). Note that $X(B_0) \leq 2.5$. Let B be a superoptimal branch not equal to B_0. By Lemma 2, B has a primary candidate u. By definition, u can receive a pebble from B. Thus $u \notin B_0$ since every element of B_0 has deficiency zero. By Lemma 4, u is an unshared primary goal of B. Thus $X(u) = -1.5$. Hence, $X(u) + X(B) \leq 0$ unless $X(B) > 1.5$. But then B is isomorphic to one of (1a)-(1f). Thus $\mu(B) = 0$. By Lemma 3, B has an unshared secondary goal u'. Since $X(u') = -1.5$ and $X(B) \leq 2.5$, we find that in all cases the sum of the excesses of B and B's goals is nonpositive. Since all the goals are unshared, $X(\mathcal{F} \cup \bigcup_{B \in \mathcal{S}} V(B)) \leq X(B_0) \leq 2.5$, as desired.

So we may assume that $\mu(B_0) \geq 1$. Suppose that $\mu(w_0) = 0$. Hence B_0 is isomorphic to one of (2a), (2b), (2d), (2e), or (2f), while each of the other elements in $\mathcal{S}$ is isomorphic to one of (2a)-(2f). Thus $X(B) \leq 1.5$ for all $B \in \mathcal{S}$. Let B be a superoptimal branch not equal to B_0. By Lemma 2, B has a primary candidate u. By definition, u can receive a pebble from B. By Lemma 4, u is an unshared primary goal of B unless $u \in B_0$. If $u \notin B_0$ then we are finished because $X(u) + X(B) \leq 0$ which implies that all the goals except for B_0 are unshared and we obtain $X(\mathcal{F} \cup \bigcup_{B \in \mathcal{S}} V(B)) \leq X(B_0) \leq 2.5$, as desired.

Consider the case when $u \in B_0$. Since B_0 is isomorphic to one of (2a), (2b), (2d), (2e), or (2f), it follows that the vertex u is unique because, for each of (2a), (2b), (2d), (2e), and (2f), there is exactly one vertex with positive deficiency. As a result, amongst the set of all branches $B \in \mathcal{S}$ not equal to B_0, there can be at most one branch, call it B_1, that does not have an unshared primary goal. So u_1 is the primary candidate of B_1. If $B \notin \{B_0, B_1\}$, $X(B)$ plus the excess of the goal associated with B is at most zero. As a result, $X(\mathcal{F} \cup \bigcup_{B \in \mathcal{S}} V(B)) \leq X(B_0) + X(B_1)$. This is at most 2.5 unless both B_0 and B_1 are both isomorphic to (2a).

Suppose that B_0 and B_1 are isomorphic to $(2a)$. We claim that B_1 has a secondary goal. Note that w_0 is the non-base '0' of B_0, because it is the only vertex with non-zero deficiency. Let v be the vertex in B_1 with six pebbles and let u_1 be the vertex in B_0 with six pebbles. Vertex u_1 is the primary candidate of B_1 and so must be at most distance two from B_1 so that it can send a pebble to it. Notice that u_1 must be distance three from the two vertices with zero pebbles on them in B_1. Otherwise, at least one pebble could be placed on one of the vertices with initially zero pebbles on B_1 and a pebble can be placed on the root with the help of the six pebbles on v. As a result, there must be a path of length two from u_1 to B_1 and this path must attach to B_1 at v. This path includes an additional vertex yet to be named. Call this vertex v_2. Note that v_2 is not in B_0 else v could send three pebbles onto B_0, a contradiction. Furthermore, u_2 cannot receive a pebble from its branch as then u_2 could obtain four pebbles, one from its branch and three from v. Then u_2 could send a pebble to w_0, a contradiction. Thus u_2 is a secondary goal and $X(B_1) + X(u_2) \leq 0$. Hence, $X(\mathcal{F} \cup \bigcup_{B \in \mathcal{S}} V(B)) \leq X(B_0) \leq 1.5$, as desired.

Lastly suppose that $\mu(w_0) = 1$. In this case, all of the elements of $\mathcal{S}$ are isomorphic to (2c). Thus $X(B) \leq .5$ for all $B \in \mathcal{S}$. Recall that a goal has excess -1.5. Hence, $X(\mathcal{F} \cup \bigcup_{B \in \mathcal{S}} V(B)) \leq .5|\mathcal{S}| - 1.5|\mathcal{F}|$. By Lemma 5, this is at most $.5(|\mathcal{F}|+5) - 1.5|\mathcal{F}| \leq 2.5$, as desired. □

6 Asymptotic Results

As a way to illustrate the techniques that are used to prove an asymptotic bound on the pebbling number for graphs of diameter four, we improve the $\mathcal{O}(\sqrt{n})$ term of Bukh's result to $\mathcal{O}(1)$. The general bound we obtain has been recently improved by Postle [10], but we include it here as it briefly illustrates our technique for the diameter four case, which still is better than Postle's bound. Recall that a vertex is heavy if $p(v) \geq 2^{\lceil \frac{d}{2} \rceil}$, where d is the diameter of the graph. Let H be the set of heavy vertices. The excess of a vertex $v \in G$ is defined by $X(v) = p(v) - 2^{\lceil \frac{d}{2} \rceil} + 1$. A vertex v is *tight* if $X(v) = 0$. Let T denote the set of tight vertices. Also let $N^k(v)$ denote the set of vertices of distance at most k from v, and $\partial N^k(v)$ denote the vertices of distance exactly k from v.

Theorem 7. *Fix d. Then $f(n, d) \leq (2^{\lceil \frac{d}{2} \rceil} - 1)n + 2^{4d} + 1$.*

Proof. Let (G, r) be a rooted graph with impotent pebbling configuration p. Our proof is broken up into two parts, depending upon the size of H.

Suppose $|H| \leq 2^{3d} + 2^{2d}$, then we can bound the total excess of G by 2^{4d}. Since p is impotent, $X(u) < 2^d - 2^{\lceil \frac{d}{2} \rceil}$ for any vertex $u \in H$. It follows that

$$\sum_{v \in V(G), X(v) > 0} X(v) \leq |H|(2^d - 2^{\lceil \frac{d}{2} \rceil}) \leq 2^{4d}.$$

So our bound holds with constant 2^{4d}, as nonheavy vertices v satisfy $p(v) \leq 2^{\lceil \frac{d}{2} \rceil} - 1$.

So we may assume that $|H| > 2^{3d} + 2^{2d}$. We will apply a discharging argument on the vertices of G. Let the initial charge of each vertex be $X(v)$ and consider the following discharging rule. For all vertices with charge greater than zero, remove charge $X(v) + 1$ and distribute this amount uniformly over $C_v = \{u \in V(G) : u \in N^{\lceil \frac{d}{2} \rceil}(v), X(u) \neq 0\}$. We now prove two claims which show that after discharging, each vertex has non-positive charge. Since the total charge in the graph does not change and the sum of the charge on each vertex is equal to the sum of the excess over all vertices of G, then the two claims show that number of pebbles initially on $V(G)$ is bounded above by $(2^{\lceil \frac{d}{2} \rceil} - 1)n$.

Claim 1. *Each vertex $v \in V(G)$ receives charge from at most 2^d heavy vertices.*

Proof. Define $R_v = \{u \in V(G) : u \in N^{\lceil \frac{d}{2} \rceil}(v), X(u) \geq 1\}$. Observe that each vertex in R_v can send a pebble to v. If $|R_v| \geq 2^d$, this would allow v to send a pebble to the root, as v could receive 2^d pebbles, contrary to the impotence of p. ♢

Claim 2. *Assume $|H| > 2^{3d} + 2^{2d}$. For any vertex v with $X(v) \neq 0$, the charge received from any heavy vertex u in $N^{\lceil \frac{d}{2} \rceil}(v)$ is at most $\frac{1}{2^d}$.*

Notice that proving this claim suffices to prove the theorem because then each non-tight vertex receives at most $2^d \cdot \frac{1}{2^d} = 1$ unit of total charge.

Proof. Let τ be a spanning BFS tree rooted at some arbitrary $v \in V(G)$. Define $A_v = \{u \in V(G) : u \in \partial N^{\lceil \frac{d}{2} \rceil}(v), X(u) \neq 0, \text{ and } u \text{ is an ancestor of some } w \in H \text{ in } \tau\}$. We wish to account for the location of all the vertices in H so that we can bound $|A_v|$. Notice that there are at most 2^d heavy vertices in $N^{\lceil \frac{d}{2} \rceil}(v)$, as otherwise v could receive 2^d pebbles. Also, there are at most $2^d - 1$ tight vertices in $\partial N^{\lceil \frac{d}{2} \rceil}(v)$ which have descendants in H, as each tight vertex can be made heavy by making pebbling moves from its heavy descendants in τ. Furthermore, each tight vertex can have at most 2^d heavy descendants in τ. (Recall that the descendants of the tight vertex can be distance at most $2^{\lfloor d/2 \rfloor}$ away, as the graph is diameter d and τ is a BFS tree.)

So far we have determined the position of at most 2^{2d} vertices in H. Now, all the other vertices in H must have a unique ancestor in τ from A_v. Moreover, each vertex in A_v is the ancestor in τ of at most 2^d vertices in H. Therefore $|A_v| \geq \frac{|H| - 2^{2d}}{2^d}$. By our assumption on the size of H,

$$|A_v| \geq \frac{|H| - 2^{2d}}{2^d} > 2^{2d}.$$

As $A_v \subseteq C_v$, we see that $|C_v| > 2^{2d}$. In particular this holds for all $v \in H$, which suffices to prove the claim. ◊

Thus, $f(n,d) \leq (2^{\lceil \frac{d}{2} \rceil} - 1)n + 2^{4d} + 1$, as desired. □

7 Diameter Four Results

We will now prove an asymptotic result that gives a tighter bound for the case when $d = 4$. In particular, we will show that $f(n,4) = \frac{3}{2}n + \Theta(1)$. Notice that this bound beats $3n + \mathcal{O}(1)$, the bound implied by Theorem 7. For this proof, the definition of excess of a set S of vertices returns to $X(S) = \sum_{v \in S}(p(v) - 1.5)$. Before we prove the asymptotic result, we require the following lemma.

Lemma 6. *Let (G,r) be a rooted graph of diameter at most four with pebbling configuration p and associated BFS tree τ rooted at r. Partition G into irreducible branches using a τ-marking, and let B be an irreducible branch with no heavy vertices. The following statements hold:*

1. *If the pebbling capacity of B is zero, then $X(B) \leq 0$.*
2. *Suppose the pebbling capacity of B is one. Suppose further that, if an additional pebble can be placed at the base of B via pebbling moves outside of B, then B still has no heavy vertex. Then $X(B) \leq 1$.*
3. *If the pebbling capacity of B is one, then $X(B) \leq 1.5$.*

Proof. We proceed via induction on the number of vertices in B.

We will first show the base case. If there is only one vertex in B, say b, then for the pebbling capacity of B to be zero, $p(b) \leq 1$, thus verifying the conclusion of statement (i). If the pebbling capacity of B is one, to satisfy the assumption of statement (ii), $p(b) \leq 2$ and to satisfy the assumptions of statement (iii), $p(b) \leq 3$. This satisfies the excess constraints for the conclusions of statements (ii) and (iii).

Now, suppose that branch B, with base b, satisfies the hypothesis of statement (i). If $p(b) = 1$ then $B = \{b\}$, as any vertices below b must send an additional pebble to b, contradicting the assumption that B had pebbling capacity zero. Thus, we are finished by the base case. If $p(b) = 0$, then b can have one subbranch B_1 of pebbling capacity at most one. Observe that B_1 has pebbling capacity at most one, as it is assumed that there are no heavy vertices in B. (Instead of repeating this argument many times in the remainder of this proof, we will implicitly assume it to be the case for any branch with the same properties as B_1.) By our induction hypothesis, $X(B_1) \leq 1.5$, so the total excess of B is at most 1.5 - 1.5 = 0, as desired. This proves statement (i).

Next, suppose that branch B, with base b, satisfies the hypotheses of statement (ii). Notice that $p(b) \leq 2$, as otherwise b would contradict the hypothesis of statement (ii). If $p(b) = 2$, then if $B = \{b\}$, we are finished. Otherwise there could be one subbranch, B_1, adjacent to b. However, its pebbling capacity must be zero, as otherwise b could become a heavy vertex when another pebble from outside B is placed on it. Thus the total excess of B is at most .5. If $p(b) = 1$, then B can only have one subbranch attached to b (if there were more, then b could become a heavy vertex with an additional pebble), and this subbranch, call it B_1, has pebbling capacity at most 1. Then the total excess of B is at most -.5 + 1.5 = 1, satisfying condition (ii). Finally, if $p(b) = 0$, then notice that B can't have three subbranches. Otherwise b, with an additional pebble added to it, could become a heavy vertex. If B has only one subbranch, then B would not have capacity one. Thus B has two subbranches below b, called B_1 and B_2. Observe that both of these two subbranches must also satisfy the hypothesis of statement (ii). Otherwise, if say B_1 did not satisfy the hypothesis of statement (ii) we could take a pair of pebbles from the base of B_2, send one pebble to b and then with the additional pebble placed on B, send a pebble to the base of B_1. As a result of this observation, the maximum excess of B_1 and B_2 is one. Since the excess of $\{b\}$ is -1.5, the conclusion of statement (ii) is satisfied as 2(1) - 1.5 = .5.

Finally, suppose that branch B, with base b, satisfies the hypotheses of statement (iii). If $p(b) = 3$, then $B = \{b\}$ and we are finished. If $p(b) = 2$, then there can be one subbranch, B_1, adjacent to b. This subbranch must satisfy the hypothesis of statement (ii) as b could send a pebble to the base of B_1. The maximum excess is therefore at most .5 + 1 = 1.5, which satisfies the conclusion of statement (iii). Suppose $p(b) = 1$. If there is one subbranch, B_1, adjacent to b, then the total excess of B is at most -.5 + 1.5 = 1, which satisfies the conclusion of statement (iii). Otherwise there are two subbranches adjacent to b, B_1 and B_2, each of which must satisfy the hypothesis of statement (ii), as we can send a pebble through b down to their respective bases. The maximum total excess of B is at most -.5 + 1 + 1 = 1.5, as desired. Finally, suppose $p(b) = 0$. In this case there can be up to three subbranches B_1, B_2, B_3 adjacent to b. If there are three subbranches, then all must satisfy the hypothesis of statement (ii), and the total excess is -1.5 + 3(1) = 1.5. If there are fewer subbranches, the total excess is at most is -1.5 + 2(1.5) = 1.5, and if there is one subbranch, the total excess is -1.5 + 1.5 = 0. In all these cases $X(B) \leq 1.5$, and so we have proved the lemma. □

Corollary 1. *Every superoptimal irreducible branch of diameter at most four with pebbling capacity zero has a heavy vertex.*

Proof. The statement of the corollary is the contrapositive to part (i) of Lemma 6. □

Remark 5. The corollary does not hold for graphs of diameter greater than four. A counterexample for $d = 5$ is the branch $0-2-0-(0-7, 0-7, 0-7)$. This is a superoptimal irreducible branch of pebbling capacity zero with no heavy vertex. Notice that this counterexample extends for $d \geq 5$. In particular, if the three paths are extended to length $\lceil \frac{d}{2} \rceil - 1$ and we place $2^{\lceil \frac{d}{2} \rceil + 1} - 1$ pebbles on each end vertex, this gives a counterexample for odd d where $d > 5$. Extending one path by an additional edge produces an analogous bound for even d.

For convenience, we introduce one more piece of terminology. For any branch B and vertex $v \in V(B)$, let $B[v]$ be the subbranch of B induced by v and its descendants in B.

Proposition 2. *If B is a capacity-one, irreducible branch of depth d with a maximum number of vertices, then all $v \in B$ have at most three children.*

Proof. If $d = 1$, by definition the base is unique. We proceed by induction on d. Suppose that B is a capacity one, irreducible branch with depth $d \geq 2$, a maximum number of vertices, and base w. By irreducibility and the fact that B is capacity one, it is clear that w has at most three children. For all $v \in B_w \setminus \{w\}$, the irreducibility of B implies that $B[v]$ is also irreducible, and the maximality of B implies that $B[v]$ is maximal as well. Hence, we are done by the inductive hypothesis as long as all $B[v]$ have capacity one. So we may assume, for purposes of contradiction, that there exists some vertex $v \in B_w \setminus \{w\}$ such that $B[v]$ has pebbling capacity greater than 1. Further, choose v such that its distance from r is maximal. Thus, for all v' that are descendants of v in B, $B[v']$ has capacity exactly 1.

Let u be the parent of v in B. Consider two cases depending upon $p(v)$. First assume $p(v) = 0$. Then v must have at least four children, say c_1, c_2, c_3, c_4. In this case, add a new vertex v' to the graph as follows. Make v' adjacent to c_1, c_2 and u. Remove the edges c_1v and c_2v. Let $p(v') = 0$. This creates a capacity one branch with depth d and more vertices than B, a contradiction.

Second, assume that $p(v) > 0$. If $d(v, w) = d - 1$ then $p(v) \geq 4$. If u does not have two additional children v_1 and v_2, expand B by setting $p(v) = 2$ and adding a new vertex e which is adjacent to u, and where $p(e) = 2$. Thus, we have created a capacity one branch of depth d with more vertices than B, a contradiction. If instead v_1 and v_2 are additional children of u, then expand B by adding a new vertex v' as follows. Let v' be adjacent to v and delete the edge uv. Make v' adjacent to the parent of u, and set $p(v') = 0$. Again, we have created a capacity one branch of depth d with more vertices than B, a contradiction.

If $p(v) > 0$ and $d(v, w) < d - 1$, expand B as follows. Remove a pebble from v and give v an additional child, v'', where $p(v'') = 2$. This again contradicts the maximality of B. □

With this in mind we can prove the following corollary.

Corollary 2. *If B is an irreducible branch of depth d with pebbling capacity zero, then* $|V(B)| \leq 1 + 1 + 3 + \cdots + 3^{d-2} = \frac{3^{d-1}+1}{2}$.

Proof. If $d \geq 1$, then since B has capacity zero, the base vertex w has one child, w'. We can then apply the previous proposition to this capacity one subbranch. □

One useful consequence of this corollary is that any irreducible, superoptimal, zero capacity branch of depth at most four has at most 14 vertices.

Lemma 7. *Suppose that B is a superoptimal, irreducible, zero-capacity branch of depth four, with no heavy vertex u in which $d(u) = 4$. Then it has no vertices with three pebbles at depth four.*

Proof. Suppose, for purposes of contradiction, that there exists a branch B satisfying the conditions of the lemma with a vertex u which satisfies $p(u) = 3$ and $d(u) = 4$. Let v be the parent of u in B. For any $w \in B$, recall that $p_B^*(w)$ defines the maximum number of pebbles w can obtain from pebbling moves in B. Note that since u is not heavy, $p_B^*(u) = 3$.

First we claim that $p_B^*(v) = 2$. As B is irreducible, $B[v]$ has nonzero capacity. That is, v can obtain at least two pebbles by pebbling moves in $B[v]$. So $p_B^*(v) \geq 2$. However, u can send at most one pebble to v. Thus if v could obtain at least three pebbles from pebbling moves in B, then v could obtain at least two pebbles by pebbling moves in

$B \setminus \{u\}$. But then v could send a pebble to u, a contradiction. So $p^*_B(v) \leq 2$. This implies that $B[v]$ has capacity one.

Since B has zero capacity, the base of B has precisely one child, call it x. Thus x is the parent of v. We claim that $p^*_B(x) = 2$. As B is irreducible, $B[x]$ has nonzero capacity. So $p^*_B(x) \geq 2$. However, v can send at most one pebble to x as $B[v]$ has capacity one. Thus if x could obtain at least three pebbles from pebbling moves in B, then x could obtain at least two pebbles by pebbling moves in $B \setminus B[v]$. But then x could send a pebble to v, which would imply that $p^*_B(v) \geq 3$, a contradiction. So $p^*_B(x) \leq 2$.

As B is an irreducible, superoptimal, zero-capacity branch, by Corollary 1, B has a heavy vertex, call it z. As B is zero capacity, z must have depth at least three. Yet by hypothesis, z cannot be at depth four. Thus z must be a child of x. Since z is heavy, $p^*_B(z) \geq 4$. Thus, $z \neq v$. Since B is irreducible, $B[z]$ has nonzero pebbling capacity, and hence x can receive a pebble from z. Therefore, x must not be able to send a pebble to z by pebbling moves in $B \setminus B[z]$, since $p^*_B(x) = 2$. This implies, since $p^*_B(z) \geq 4$, that z can obtain at least four pebbles from pebbling moves in $B[z]$. But then $B[z]$ has capacity at least two. Thus x can receive at least two pebbles from z and one from v, implying that $p^*_B(x) \geq 3$, a contradiction. □

In the proof of the main theorem in this section, Theorem 8, we require two constants: let b_c, the *branch constant*, be the maximum number of vertices per irreducible, superoptimal, zero-capacity branch, and let p_c, the *pebbling constant*, be the maximum number of pebbles per vertex. By Corollary 2, $b_c = 14$. Since G is diameter four, $p_c = 15$.

Theorem 8. *Let G be a graph of diameter four on n vertices. Then $\pi(G) \leq \frac{3n}{2} + (7 \cdot 14^5 \cdot 15^5)$. Furthermore, $f(n,4) = \frac{3n}{2} + \Theta(1)$.*

Proof. Let $r \in V(G)$ and p be an impotent pebbling configuration for (G, r). We will show that $\sum_v p(v) \leq 3n/2 + (7 \cdot 14^5 \cdot 15^5)$. Combining this with Theorem 5 yields the theorem.

Let τ be a BFS tree rooted at r such that the τ-marking M of G, whose existence is guaranteed by Proposition 1, satisfies the following conditions:

1. The number of branches of depth four in M is maximized.
2. Subject to condition 1, the number of vertices $v \in V(G)$ such that $d(v) = 4$, $p(v) = 3$ and B_v in M has depth four, is maximized.

It suffices to prove that $\sum_{B \in M} X(B) = \mathcal{O}(1)$. Let $\mathcal{S}$ be the set of branches B in M such that $X(B) \geq 0$. Recall that by Corollary 1, every superoptimal branch has a heavy vertex. For every branch B in $\mathcal{S}$, let v_B be a heavy vertex in B of maximal distance from the base of B. Call this the *representative heavy vertex* of B. Let $\mathcal{H}$ be the set of all representative heavy vertices.

If $|\mathcal{H}| < 7b_c^4 p_c^4$, then there are fewer than $7b_c^4 p_c^4$ superoptimal branches in M. Since for every branch B in M, $X(B) \leq b_c p_c$, $\sum_{B \in M} X(B) \leq (7b_c^4 p_c^4) b_c p_c$ and the theorem holds.

So we may assume $|\mathcal{H}| \geq 7b_c^4 p_c^4$. Now we use a discharging argument to show that $\sum_{B \in M} X(B) \leq 0$. For every branch B in M, let the initial charge of B be equal to $X(B)$. Let T be the set of all branches B in M such that B does not have a heavy vertex and $X(B) = 0$. Let $V(T)$ be the set of all vertices that are contained in branches in T. Now we discharge the branches in $\mathcal{S}$ according to the following rule: let B be a branch in $\mathcal{S}$ and let $v = v_B$ be its representative heavy vertex. Distribute $X(B) + .5$ units of charge from B uniformly over all branches not in T that intersect $N^2(v)$, including itself.

We will show that the final charge of each branch is non-positive. To prove this we need the following two claims.

Claim 3. *Every branch in $M \setminus T$ receives charge from at most $b_c p_c$ branches.*

Proof. Let B be a branch in $M \setminus T$. Suppose that B receives charge from more than $b_c p_c$ branches. Then by the pigeonhole principle there exists a vertex $u \in B$ that is in the second-neighborhood of p_c representative heavy vertices (in fact, we get p_c+1). Since each heavy vertex can obtain four pebbles from its own branch, they can each simultaneously send a pebble to u. Now u has p_c pebbles and can therefore send a pebble to r, a contradiction since the pebbling configuration was assumed to be impotent. $\diamondsuit$

Claim 4. *If $|\mathcal{H}| > 7b_c^4 p_c^4$, then every branch in $\mathcal{S}$ sends at most $\frac{1}{2b_c p_c}$ units of charge to any branch in M.*

Proof. Let B be a branch in $\mathcal{S}$ with representative heavy vertex $v = v_B$. Note that $X(B) + .5 < b_c p_c$, as otherwise there would exist a vertex u in B with at least p_c pebbles. But then u could send a pebble to r, a contradiction. Since B discharges $X(B) + .5$ units of charge uniformly to all branches not in T that intersect $N^2(v)$, it suffices to show that there are at least $2b_c^2 p_c^2$ branches in $M \setminus T$ intersecting $N^2(v)$. To show this, it is enough to show that there are at least $2b_c^3 p_c^2$ vertices in $N^2(v) \setminus V(T)$.

Let κ be a BFS tree of G rooted at v. Let $A = \{u \in V(G) | u \in \partial N^2(v)$, and u is the ancestor in κ of some vertex in $\mathcal{H}\}$. We claim that $|A| \geq 6b_c^4 p_c^3$. To see this, notice that there are at most p_c representative heavy vertices in $N^2(v)$, as otherwise v could receive p_c pebbles, a contradiction. Hence there are least $|\mathcal{H}| - p_c$ vertices in $\mathcal{H}$ that are not in $N^2(v)$. These must be descendants in κ of vertices in A. However, every vertex in A has at most p_c descendants in $\mathcal{H}$, as otherwise such a vertex could receive at least p_c pebbles, a contradiction. Thus, $|A| \geq \frac{|\mathcal{H}| - p_c}{p_c} > \frac{7b_c^4 p_c^4 - p_c}{p_c} \geq 6b_c^4 p_c^4 / p_c = 6b_c^4 p_c^3$.

If $|A \setminus V(T)| \geq 2b_c^3 p_c^2$, then Claim 4 follows as noted above. So we may assume that $|A \setminus V(T)| < 2b_c^3 p_c^2$. This implies that $|A \cap V(T)| > 6b_c^4 p_c^3 - 2b_c^3 p_c^2 \geq 6b_c^4 p_c^3 - 2b_c^4 p_c^3 = 4b_c^4 p_c^3$. For each branch in T that intersects A, pick a vertex in $A \cap V(T)$. Let R be the set of all such vertices. Note that $|R| \geq |A \cap V(T)| / b_c \geq 4b_c^4 p_c^3 / b_c = 4b_c^3 p_c^3$.

Let $C = \{u \in N(v) | u$ is the parent in κ of some $w \in R\}$. Note that $|C| \geq |R|/p_c \geq 4b_c^3 p_c^3 / p_c = 4b_c^3 p_c^2$, as otherwise there exists a vertex u in C with at least p_c children in R. Each such vertex can simultaneously receive two pebbles, one from its own branch and one from its descendant in $\mathcal{H}$. Thus u could receive p_c pebbles, a contradiction. Furthermore, note that if $|C \setminus V(T)| \geq 2b_c^3 p_c^2$, then $|N^2(v) \setminus V(T)| \geq 2b_c^3 p_c^2$ as $C \subseteq N^2(v)$ and Claim 4 follows as noted above. Thus we may assume that $|C \setminus V(T)| < 2b_c^3 p_c^2$. Hence $|C \cap V(T)| > 4b_c^3 p_c^2 - 2b_c^3 p_c^2 = 2b_c^3 p_c^2$.

Let $D = \{u \in C \cap V(T) | u$ is the base of B_u in $M\}$. We claim that $|(C \cap V(T)) \setminus D| \leq b_c p_c$. Suppose not. Then there are least p_c vertices in $N(v)$ which are from distinct branches in T and not the base of their own branch. If a vertex is not the base of its own branch, it can obtain at least two pebbles from its branch. Thus each of these vertices can send a pebble to v, a contradiction. This claim implies that $|D| > 2b_c^3 p_c^2 - b_c p_c \geq 2b_c^3 p_c^2 - b_c^3 p_c^2 = b_c^3 p_c^2$.

Since v is a heavy vertex, $d(v) \geq 3$, as otherwise v could send a pebble to r, a contradiction. Thus, if $u \in D$, we find $d(u, r) \geq 2$, as u is adjacent to v. Moreover, since u is the base of B_u by definition of the set D, B_u has depth at most three.

Now we characterize B_u for $u \in D$. We claim that B_u is of the form $0 - 3$ or $0 - 0 - (3, 3)$. First, observe that the number of vertices in B_u is even as the number of pebbles on B_u is $\frac{3}{2}|V(B_u)|$ (recall that $B_u \in T$ implies $X(B_u) = 0$), which must be an integer. This implies that B_u does not have depth one. Recall from the proof of Corollary 2, that there is at most one vertex adjacent to the base of a branch. Thus, if B_u has depth two, B_u must contain exactly two vertices, and hence B_u is 0-3. Since B_u has no heavy vertices, there can then be at most three vertices distance two from the base of B_u. Thus if B_u is depth three, B_u must contain exactly four vertices and six pebbles. The only way to distribute the pebbles so that there are no heavy vertices is $0 - 0 - (3, 3)$.

We claim that B_u has depth at most two, and thus by the paragraph above is of the form $0 - 3$. Suppose not. Then B_u has depth three and B_u is of the form $0 - 0 - (3, 3)$. Hence $d(u, r) = 2$ (we showed above that $d(u, r) \geq 2$) and $d(v, r) = 3$. Let w be the parent of v in τ. Consider the BFS tree τ' rooted at r, where $\tau' = (\tau \cup \{uv\}) \setminus \{wv\}$. By the properties of irreducibility, the subbranch induced by v and its descendants has nonzero capacity. We claim that that the branch B'_v containing v in the τ'-marking M' of G has depth four. To see this, notice that u can receive a pebble from B_u and, since v was not a base vertex in τ, it must be able to send a pebble from $B[v]$ to u. As a result, $p_{B'_v}(u) \geq 2$. Therefore u is not the base of B'_v. However, u was the base of B_u, a depth three branch. Since B_u is contained in B'_v, we find that B'_v must have depth four.

If B_v in τ has depth at most three, then M' has more branches of depth four than M, contradicting the choice of τ. So we may assume that B_v has depth four. Now M' has at least as many branches of depth four as M. However, since v is the representative heavy vertex of B_v, this implies that B_v has no heavy vertices of depth four. By Lemma 7, B_v has no vertices with three pebbles at depth four. Yet, B'_v also contains the vertices in B_u. Thus there are at least two vertices with three pebbles at distance four from the root that were not in a branch of depth four in M but are in a branch of depth four in M'. Thus τ' contradicts the choice of τ, and thus proving the claim that B_u has depth at most two.

For each $w \in R$ without a pebble, make pebbling moves in B_w to place a pebble on w. Then for each $u \in D$, remove two pebbles from the vertex with three pebbles in B_u to send a pebble to u. Now each vertex in D has one pebble and each vertex in R has at least one pebble. Each vertex u in D has a child w in R which has a descendant z in $\mathcal{H}$. Make pebbling moves in B_z so that z obtains four pebbles. Then use these to send a pebble to w. Now w has two pebbles. Use these to send a pebble to u. Now u has two pebbles. Use these pebbles to send a pebble to v. However, since we have shown $|D| > b_c^3 p_c^2$, we find that v can obtain at least p_c pebbles, a contradiction. ◊

Let B be a branch in M. We will show that the final charge of B is non-positive. First suppose that B is in T. Since the initial charge of B is zero, and since B does not receive charge from any other branch, the final charge of B is zero. Second, suppose that B is in $\mathcal{S}$. The initial charge of B is $X(B)$. According to the discharging rules, B sends $X(B) + .5$ units of charge to other branches. From Claims 3 and 4 above, B receives at most .5 units of charge from other branches. Hence the final charge of B is at most $X(B) - (X(B) + .5) + .5 = 0$. Finally, suppose that $X(B) < 0$. Recall that $X(B) = \sum_{v \in B}(p(v)) - 1.5|B|$. Since $p(v)$ is integral for all v and $|B|$ is also integral, this implies that $X(B) \leq -.5$. As B receives at most .5 units of charge from other branches, the final charge of B is at most zero. □

References

[1] B Bukh. Maximum pebbling number of graphs of diameter three. *J. Graph Theory* **52** (2006), 353–357.

[2] F Chung. Pebbling in hypercubes. *SIAM J. Discrete Mathematics* **2** (1989), 467–472.

[3] T A Clarke, R A Hochberg, and G H Hurlbert. Pebbling in diameter two graphs and products of paths. *J. Graph Theory* **25** (1997), 119–128.

[4] P Erdős, A Ginzburg, A Ziv. A theorem in additive number theory. *Bull. Res. Council Isreal* **10F** (1961) 41–43.

[5] D S Herscovici, B D Hester, G H Hurlbert. Diameter bounds, fractional pebbling and pebbling with arbitrary target distributions, `arXiv:0905.3949v1`[math.CO] (2009).

[6] G Hurlbert. A survey of graph pebbling. *Congr. Numer.* **139** (1999), 41–64.

[7] G Hurlbert. Recent progress in graph pebbling. *Graph Theory Notes of New York* **XLIX**, (2005), 25–37.

[8] P Lemke, D Kleitman. An addition theorem on the integers modulo n. *J. Number Theory* **31** (1989) 335–345.

[9] L Pachter, H S Snevily, and B Voxman. On pebbling graphs. *Proccedings of the Twenty-Sixth Southeastern International Conference on Combinatorics, Graph Theory and Computing* (Boca Raton, FL, 1995) **107** (1995), 65–80.

[10] L Postle. Pebbling graphs of fixed diameter. Submitted, (2011).