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Optimal Pebbling on Grids

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Abstract Given a distribution of pebbles on the vertices of a connected graph G, a *pebbling move* is defined as the removal of two pebbles from some vertex and the placement of one of these on an adjacent vertex. The *pebbling number* of a graph G is the smallest integer k such that for each vertex v and each distribution of k pebbles on G there is a sequence of pebbling moves that places at least one pebble on v. We say such a distribution is *solvable*. The *optimal pebbling number* of G, denoted $\Pi_{OPT}(G)$, is the least k such that some particular distribution of k pebbles is solvable. In this paper, we strengthen a result of Bunde et al relating to the optimal pebbling number of the 2 by n square grid by describing all possible optimal configurations. We find the optimal pebbling number for the 3 by n grid and related structures. Finally, we give a bound for the analogue of this question for the infinite square grid.

Keywords pebbling · optimal · fractional · grid

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1 Introduction

A recent development in graph theory, suggested by Lagarias and Saks (via a private communication to Chung), is called *pebbling*. Pebbling was first introduced into the literature by Chung who computed the pebbling number of Cartesian products of paths to give a combinatorial proof of the following number-theoretic statement of Kleitman and Lemke.

Theorem 1 *[2][8] Let $\mathbb{Z}_n$ be the cyclic group on n elements and let $|g|$ denote the order of a group element $g \in \mathbb{Z}_n$. For every sequence $g_1, g_2, \ldots, g_n$ of (not necessarily distinct) elements of $\mathbb{Z}_n$, there exists a zero-sum subsequence $(g_k)_{k \in K}$, such that $\sum_{k \in K} \frac{1}{|g_k|} \leq 1$. Here K is the set of indices of the elements in the subsequence.*

Chung developed the pebbling game to give a more natural proof of this theorem. Theorems of this type play an important role in this area of number theory as they generalize zero-sum theorems such as the Erdős-Ginzburg-Ziv [3] theorem. Over the last twenty-five years, pebbling has developed into its own subfield with over sixty papers. Hurlbert has written two survey papers [6] [7], that outline the history of pebbling in more detail.

Given a connected graph G, distribute *pebbles* (indistinguishable markers) on its vertices. Initially, each vertex is assigned a certain amount of pebbles according to a *distribution* D, which is a function $D : V(G) \to \mathbb{N} \cup \{0\}$. A *pebbling move* from a vertex v to an adjacent vertex u takes away two pebbles at v and adds only one pebble to u. A *pebbling sequence* is a sequence of pebbling moves.

Given a distribution D, if we can put one pebble on a "root" vertex v after some pebbling moves, v is said to be *reachable* under D. If we can send some pebbles from one endpoint of an edge e to the other endpoint of e, we say that e is reachable under D. A distribution D is *solvable* if and only if all vertices of G are reachable under D. The *pebbling number* of a graph G, denoted $\Pi(G)$, is the least k ($k \in \mathbb{N}$) such that any distribution of k pebbles on G is solvable.

The *optimal pebbling number* of G, denoted $\Pi_{OPT}(G)$, is the least k such that some particular distribution of k pebbles is solvable. The optimal pebbling number was first investigated in a result of Pachter, Snevily and Voxman [11] who showed that $\Pi_{OPT}(P_n) = \lceil 2n/3 \rceil$. Later, Moews [10] showed that for the k-cube, Q_k, $(4/3)^k \leq \Pi_{OPT}(Q_k) \leq (4/3)^{k+\mathcal{O}(\log k)}$. Further, computing the optimal pebbling number is NP-hard by a result of Milans and Clark [9]. The most recent results related to optimal pebbling are those of Bunde et al [1], whose results we will describe in the next section and Friedman and Wyels [4], who compute the optimal pebbling number of paths and cycles.

2 Optimal Pebbling Results

Definition 1 Given a distribution D, a vertex is said to be *k-reachable* if we can put k pebbles onto this vertex after some pebbling sequence. If all vertices are k-reachable under D, D is *k-solvable.* We first state a result of [1] about 2-solvability on the path.

Theorem 2 *Every 2-solvable distribution on P_n has at least $n+1$ pebbles. Furthermore, the 2-solvable distributions with $n+1$ pebbles consist of "prime segments" separated by single unoccupied vertices, where a prime segment is a path with either (1) two pebbles on one vertex and one pebble on all other vertices, or (2) three consecutive vertices having 0,4,0 pebbles, respectively, and one pebble on all other vertices.*

Theorem 2 gives us a lower bound for the number of pebbles required to make a path P_n 2-solvable. In addition, it describes the configuration of an optimal distribution, which is essential to the proof of Theorem 3.

Definition 2 A graph H is a *quotient* of a graph G if the vertices of H correspond to the sets in a partition of $V(G)$, and distinct vertices of H are adjacent if at least one edge of G has endpoints in the sets corresponding to both vertices of H. In other words, each set in the partition of $V(G)$ *collapses* to a single vertex of H.

The *Collapsing Lemma* of [1] is widely used in this paper. It is especially important in the proof of Theorem 6, where we first collapse $P_m \square C_3$ into $P_m \square C_2$.

Lemma 1 *(Collapsing Lemma) Let H be a quotient of G via ϕ. If a distribution D' on G is obtainable from a distribution D on G via pebbling moves, then in H any vertex v is $D'_\phi(v)$ − reachable under D_ϕ. In particular, $\Pi_{OPT}(G) \geq \Pi_{OPT}(H)$.*

Definition 3 In $P_m \square K_2$ or $C_m \square K_2$, we call the m copies of K_2 the *rungs* of the graph. Similarly, in $P_m \square P_3$ or $C_m \square P_3$, we call the m copies of P_3 the *rungs* of the graph.

Definition 4 If some pebbling sequence from a distribution D results in a rung having at least two pebbles, we say that the rung is *2-reachable* under D. If all the rungs are *2-reachable*, then collapsing each rung to a vertex yields a path that is *2-solvable.*

Theorem 3 *[1] It holds that $\Pi_{OPT}(P_m \square K_2) = \Pi_{OPT}(C_m \square K_2) = \Pi_{OPT}(M_m) = m$ for $m \geq 2$, except that $\Pi_{OPT}(P_2 \square K_2) = \Pi_{OPT}(C_2 \square K_2) = 3$ and $\Pi_{OPT}(P_5 \square K_2) = 6$.*

In the later proofs, we might refer to the graph $P_m \square K_2$ as the *ladder* and each P_m as one *side* of the ladder.

3 New Results about $P_m \square K_2$

We begin with a few helpful definitions related to our proofs. If a vertex v is reachable under some distribution D, we say that D *covers* v or v is covered by D. Moreover, the set of all covered vertices by D is called the *coverage* of D. The *size of a distribution* D, expressed as $|D|$, is the total amount of pebbles used in D, while the *size of the coverage* by a distribution D, expressed as $Cov(D)$, is the total number of vertices covered by D. A *unit* is the coverage obtained by only considering pebbles on a single vertex, called the *source*, and ignoring other pebbles. For example, Figure 1 is a unit (indicated by the shaded region) on $P_n \square K_2$ and v is a source with four pebbles.

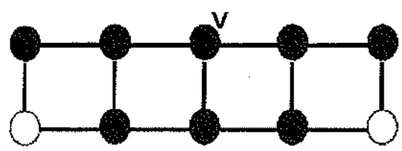

Fig. 1 A unit with four pebbles on source v.

A *block* is a combination of several units such that the subgraph induced by all reachable edges is connected. Note that a unit can also be considered as a block. For example, in Figure 2, both graphs are composed of two units (source v has four pebbles and source u has two pebbles). However, the top graph is a block while the bottom one is not because the dotted edge is not reachable.

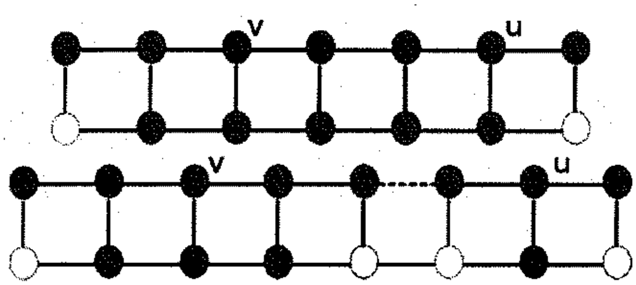

Fig. 2 Connected and unconnected subgraphs induced by reachable edges.

Since units and blocks describe coverages, the *size of a unit or a block* is defined as the number of vertices covered by that unit or block. Suppose we have two vertices u and v, the *distance* between them, denoted $dist(v, u)$ or $dist(u, v)$, is the number of edges in a shortest path that connects them. Let

u be a covered vertex by a distribution D. If u is adjacent to an uncovered edge, we say that u is on the *border* of D and we call u a *boundary vertex*.

Lemma 2 *On $P_m \square K_2$, all units of size greater than one has a symmetric "stair-like" shape. (See Figure 1.)*

Proof Suppose $2^m + k$ $(m, k \in \mathbb{Z}, m \geq 1,\ 0 \leq k < 2^m)$ pebbles are placed on the source v. If u is a boundary vertex, then $dist(v, u) = m$. Therefore, on the occupied side of the ladder, the pebbles can travel a distance of m. However, on the other side of the ladder, those pebbles can only travel a distance of $m - 1$ because the distance between the two sides is 1.

Lemma 3 *A block B on $P_m \square K_2$ can only have two types of ends: Type I (the sharp end) or Type II (the kinked end). (See Figure 3.)*

Proof Let u be a vertex on the border of the coverage, v_1 be an adjacent covered vertex that is on the same side of the ladder, and v_2 be another adjacent vertex that is on the opposite side of the ladder (not necessarily covered).

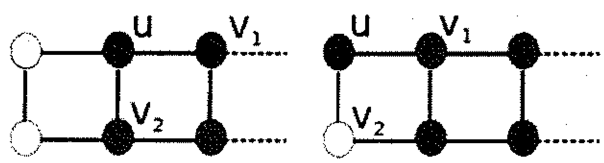

Fig. 3 Type I end (left) and Type II end (right).

Since u is on the border, u is at most 1-reachable. Otherwise, the coverage can be extended to a *new* vertex ("extend" means "cover new vertices" and "new" means "unreachable by the current distribution"). Furthermore, v_1 and v_2 are at most 3-reachable. If v_2 is 2-reachable or 3-reachable, we have a Type II end (not shown in Figure 3). If v_2 is 1-reachable, we have a Type I end. If v_2 is unreachable, we also have a Type II end because v_1 must be 2-reachable or 3-reachable.

We now state a definition that helps us describe relationships between multiple blocks. If two blocks have a vertex in common, then we say they *interact*. We call the vertices covered by both blocks *interaction vertices*. A block can be thought of as several interacting units.

Lemma 4 *On $P_m \square K_2$, if two units interact, they can cover at most four new vertices. In particular, at most two new vertices can be covered at each end of the block.*

Proof To understand this more easily, we introduce the binary system. If we write the number of pebbles on one vertex in the binary system, observe that we can find the maximum reachability of a vertex n steps away by dropping the last n digits of that binary number. Therefore, if the original vertex has m digits, the maximum steps these pebbles can travel is given by $m-1$.

Since by adding two binary numbers we can increase the number of digits by at most one, the maximal distance the pebbles on these two sources combined can travel increases by at most 1. Because the block has exactly four unoccupied neighbors (two on each end), we can cover at most two new vertices on the left end and at most two new on the right end.

If there is only one interaction vertex between the two units, then we can only cover one new vertex, which is also between them. (See Lemma 6 for more details.)

Definition 5 The *covering ratio* of a distribution D is given by

$$\frac{Cov(D)}{|D|},$$

which is the ratio between the size of the coverage and the size of the distribution.

Lemma 5 *On $P_m \square K_2$, all units have a covering ratio less than or equal to 2. Specifically, the only units that have a covering ratio of 2 are units whose sources have two or four pebbles.*

Proof Suppose a source has $2^m + k$ ($m, k \in \mathbb{N} \cup \{0\}, m \geq 0, 0 \leq k < 2^m$) pebbles. Then the size of the coverage is given by $4m$ ($2m+1$ on one side and $2m-1$ on the other). Therefore, the covering ratio of this unit is $4m/(2^m+k)$. To find the maximum of this expression, we set $k=0$ and take its derivative. The unique maximum occurs when $m = 1/ln(2) \approx 1.44$. Since $m \in \mathbb{N} \cup \{0\}$, it is either 1 or 2 or both. After computing the covering ratio, we see that both yield a covering ratio of 2.

Definition 6 Suppose we have a distribution D on a graph G. Now if we put some extra pebbles on G to obtain a new distribution D', then we say that the *marginal coverage* of these pebbles is $Cov(D') - Cov(D)$ and the *marginal covering ratio* is given by

$$\frac{Cov(D') - Cov(D)}{|D'| - |D|}.$$

Lemma 6 *On $P_m \square K_2$, the covering ratio of two interacting units is at most 2. More specifically, an interaction does not increase the covering ratio unless*

a. two sources are adjacent and each has three pebbles or,
b. one source has just one pebble.

Proof First, by Lemma 5, if two units do not interact, the covering ratio can be at most 2. Therefore, if we want to obtain a covering ratio greater than 2, these two units must interact. In order for the covering ratio to increase, we need to increase the size of the coverage without changing the number of pebbles used. This means that the number of new vertices covered due to the interaction must be strictly less than the number of interaction vertices. Suppose we have a unit U_1 whose source has x pebbles and another unit U_2 whose source has y pebbles.

Now if U_1 and U_2 have only one interaction vertex and $x, y \geq 2$, then the source must be on the same side of the ladder. Otherwise, there exist at least two interaction vertices due to the shape of a unit. Besides, there exists one unreachable vertex (from either unit) between the two units. (See Figure 4.) Hence, the covering ratio does not change after the interaction because only one new vertex (the crossed one) is covered.

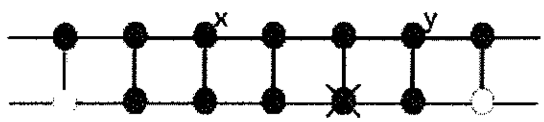

Fig. 4 Setting $x = 4$ and $y = 2$.

If one of the sources has one pebble, U_1 and U_2 has only one interaction vertex. Without loss of generality, let us assume that $x = 1$ and $y \geq 2$. The interaction can cover at most two new vertices because all vertices are of degree 3. Figure 5 gives an example when the other source has two pebbles. Thus, the marginal covering ratio U_1 is at most 2. By Lemma 5, the covering ratio of U_2 is no more than 2, so the overall covering ratio must be less than or equal to 2.

In other cases, there exist at least two interaction vertices, so in order to increase the covering ratio, at least two previously unreachable vertices become reachable after the interaction. By Lemma 4, at most two more vertices can be covered at either end. Therefore, both ends have to be extended in order to obtain a higher covering ratio. That is to say, both sources are able to reach the rung the other source belongs to after some pebbling moves. Otherwise, the ends cannot be extended.

Case 1: If both sources have four or more pebbles, there are at least four interaction vertices. If both sources are on the same rung, then we have at least

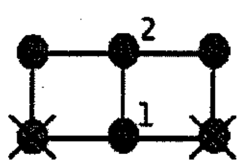

Fig. 5 The interaction between one source and another with only one pebble.

six interaction vertices. If they are on adjacent rungs, these two rungs consist of interaction vertices. If they are on non-adjacent rungs, all rungs between them consist of interaction vertices. Besides, by the argument made above, each rung with the source offers at least one interaction vertex. Thus, we have at least four interaction vertices, so the covering ratio cannot be increased.

Case 2: If one source has four or more pebbles and the other has three or less, these two sources must be on adjacent rungs. So there are at least three interaction vertices. If the two interacting units have exactly three interaction vertices, these sources must be on opposite sides on the ladder. Therefore, the source with fewer pebbles cannot reach the other source, and the coverage can be extended by at most three. (See Figure 5 for one specific example.)

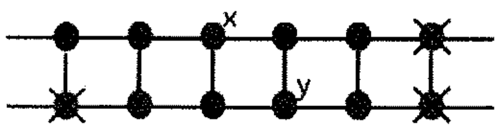

Fig. 6 Setting $x = 6$ and $y = 3$.

Case 3: If both sources have more than one but fewer than four pebbles, they must be on adjacent rungs or on the same rung. Otherwise, they have fewer than two interaction vertices. If they are on the same side of the ladder, there are two interaction vertices. If they are on opposite side of the ladder, there are two interaction vertices. Since each source has fewer than four pebble, only one pebble can be sent from one source to the other. So if a source initially has two pebbles, it can have at most three after the interaction. But the coverage is not affected by the change in the reachability of this source because it does not send more pebbles to its adjacent vertices after the interaction. Therefore, the only way to reach four new vertices is to have two adjacent sources with three pebbles. Under this circumstance, the covering ratio increases from $\frac{4}{3}$ to $\frac{5}{3}$, but is still less than 2.

Therefore, two interacting units can never yield a covering ratio greater than 2 on $P_m \square K_2$.

Lemma 7 is an extension of Lemma 6 and uses the same idea. To prove the next lemma, we also condition on the number of interaction vertices.

Lemma 7 *On $P_m \square K_2$, the covering ratio of a block is at most 2.*

Proof We will proceed by induction.
Base Case: Two units interact with each other. The covering ratio of two interacting units is at most 2 according to *Lemma 3.5.*
Inductive Hypothesis: For $n \geq 2$, suppose the covering ratio of a block that consists of n units is no more than 2. We want to show that a block that consists of $n+1$ units also has a covering ratio of 2 or less.

Analysis: Note that a block of $n + 1$ sources can be decomposed into a block with n sources and a unit. We look at the number of interaction vertices between the unit and the block.
Case 1: There is only one interaction vertex. If the source of the unit has one pebble, we can cover at most two new vertices. If the source of this unit has more than one pebble, we observe that at most one other vertex can be covered. (Reasons are addressed in Lemma 6.) Hence, the covering ratio overall is still less than or equal to 2.
Case 2: There exist two or more interaction vertices. According to Lemma 4, we can also show that at most two new vertices can be covered at each end. Therefore, to obtain a higher covering ratio, the resulting block from the interaction has to extend on both ends. This requires that the block overlaps the rung where the source of the unit is. (Otherwise, the configuration will not extend or extend only on one end.)

- The source of the unit has four or more vertices.
  If the block interacts with the unit on a sharp end, observe that there are at least five interaction vertices. If the block interacts with the unit on a kinked end, there are at least four interaction vertices. Either way, we will have at least four interaction vertices and thus not a greater covering ratio.

- The source of the unit has two or three pebbles.

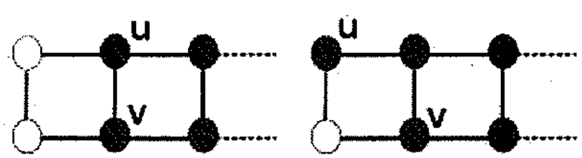

Fig. 7 Sharp end and kinked end.

Let u, v be two boundary vertices on one end of the block and suppose without loss of generality that the unit is to the left of the block.

If the block has a sharp end, the source must fall on u or v (see Figure 7) because otherwise, there will be at least four interaction vertices. Without loss of generality, let us suppose that the source is on u. If we put two pebbles on u, u becomes 3-reachable and v becomes 2-reachable, so at most two more vertices can be covered on the left end. Since previously we argued that at most two more on the right end of the block can be covered, the interaction covers at most four new vertices. The marginal covering ratio of a unit with two pebbles is at most 2. If we put three pebbles on u, u becomes 4-reachable and v becomes 2-reachable, so at most five new vertices can be covered. The marginal covering ratio of a unit with three pebbles is at most $\frac{5}{3} \leq 2$. Therefore, the covering ratio overall is at best equal to 2.

If the block has a kinked end, the source must also fall on u or v. If we have two pebbles on u, u is 3-reachable and at most two new vertices on the right end of the block can be covered. The marginal covering ratio is at most 1. If we have three pebbles on u, u becomes 4-reachable and a maximum of six new vertices can be covered. The marginal covering ratio is at most 2. If we have two pebbles on v, at most three new vertices can be covered. The marginal covering ratio is at most $\frac{3}{2}$. If we have three pebbles on v, at most five new vertices can be covered. The marginal covering ratio is at most $\frac{5}{3}$. Therefore, such interaction cannot raise the covering ratio to over 2.

Theorem 4 *We have that $\Pi_{OPT}(P_m \square K_2) \geq m$. Furthermore, in $P_m \square K_2$ ($m \neq 2, 5$), an optimal distribution D with m pebbles consists of the following two types of fundamental blocks, 2-2 blocks and 2-1 blocks. (See Figure 8.)*

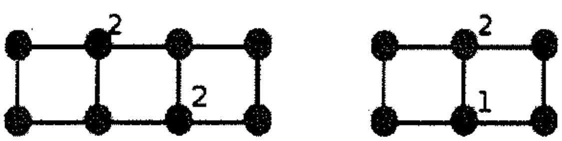

Fig. 8 A 2-2 block(left) and a 2-1 block(right).

Proof To obtain the lower bound, we apply the idea of maximum covering ratio. If a distribution D covers $P_m \square K_2$, then $Cov(D) = 2m$ and since the the covering ratio is no more than 2, $\Pi_{OPT}(P_m \square K_2) \geq m$.

If a block has one kinked end and one sharp end, the covering ratio is strictly less than two because the size of this block is odd and there is only an integer number of pebbles. Therefore, a block of covering ratio two has either two kinked ends (a kinked-ended block) or two sharp ends (a sharp-ended block). Since $P_m \square K_2$ has a sharp end, under optimal pebbling, we must put a block with two sharp ends on one end of the ladder. But if we do that, the rest of the graph becomes another shorter ladder, so we need to use another sharp-ended block. In the end, every block we use in the optimal distribution on $P_m \square K_2$ is sharp-ended.

Now let us suppose that B is a minimal sharp-ended block of covering ratio two with k sources. Suppose for a contradiction that $k \geq 3$. Then we can decompose B into a smaller block B' and a unit U. Since B is a minimal sharp-ended block, B' has to be a kinked-ended block to avoid any contradiction. (Note that in Lemma 7 we argued that the marginal covering ratio of U is at most 2, so the covering ratio of B' must be 2 in order to get an overall covering ratio of 2 after the interaction. So B' has to be either kinked-ended or sharp-ended.)

By arguments made in the proof of *Lemma 3.6*, if there is only one interaction vertex between B' and U, the resulting block B would still be a kinked-ended block, so there must be at least two interaction vertices. We have proved above that if a kinked-ended block is interacting with a unit, the only way to maintain a covering ratio of 2 is to put three or four pebbles on vertex u. But no matter which way we choose, the resulting block B is still kinked-ended, which contradicts our assumption. Therefore, if there exists a block with two sharp ends, it has exactly two sources.

By Lemma 5 and Lemma 6, the only units we can use are units with one, two, or four pebbles on their sources. We list all possible configurations and the blocks in Figure 8 are the only ones that are both sharp-ended and of covering ratio 2.

4 New Results about $C_m \square P_3$, $P_m \square C_3$, and $P_m \square P_3$

Definition 7 Define G_{inf} to be the graph with the following vertex and edge sets: $V(G_{inf}) = \{(a,b) \mid a \in \mathbb{Z}, b \in \mathbb{Z}\}$, $E(G_{inf}) = \{(a,b)(a+1,b), (a,b)(a,b+1) \mid a \in \mathbb{Z}, b \in \mathbb{Z}\}$. Informally speaking, we can think of G_{inf} as the infinite grid.

Our graphs of interest, $C_m \square P_3$ and $P_m \square P_3$, are both subgraphs of G_{inf}. In some of the case analyses in later proofs, we first show how many pebbles can be covered with a certain amount of pebbles on G_{inf}, and then apply the result to the graph we are analyzing. Because clearly, if n pebbles are reachable on G_{inf}, no more than n pebbles are reachable on any induced subgraphs of G_{inf}.

Theorem 5 *For $m \geq 3$, $\Pi_{OPT}(C_m \square P_3) = m$, except that $\Pi_{OPT}(C_3 \square P_3) = 4$.*

Proof We first prove $\Pi_{OPT}(C_3 \square P_3) = 4$ by case analysis.
Upper bound. First we show that $\Pi_{OPT}(C_3 \square P_3) \leq 4$. We can put four pebbles on a vertex in the middle row and the distribution is solvable.
Lower bound. We now show that $\Pi_{OPT}(C_3 \square P_3) \geq 4$. If we put three pebbles on three distinct vertices or on one single vertex, we cannot obtain a solvable distribution. Therefore, we must put two pebbles on a vertex and one on another. If the two vertices are adjacent to each other (Figure 9), then we can cover eight vertices (shaded dots represent vertices that can be covered). If the

two vertices are not adjacent to each other (Figure 10), then we can cover six vertices.

However, $C_3 \square P_3$ has nine vertices in total, so $\Pi_{OPT}(C_3 \square P_3) \geq 4$.

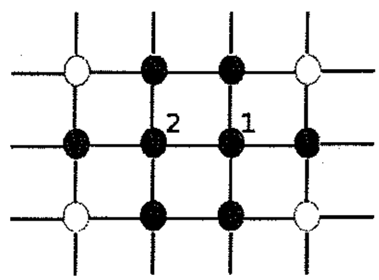

Fig. 9 Two vertices are adjacent.

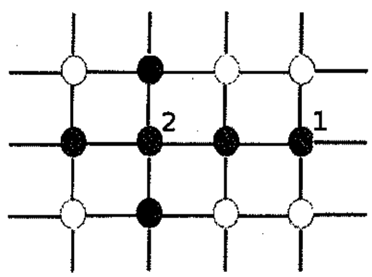

Fig. 10 Two vertices are not adjacent.

For $m > 3$, we establish the upper and lower bounds separately.

Upper bound. We show $\Pi_{OPT}(C_m \square P_3) \leq m$. If m is even, then we put two pebbles on every other vertex in the middle row and the distribution is solvable. If m is odd, we first put a pebble on some vertex in the middle row and two pebbles on one of its adjacent vertices that are also in the middle row. After that, we put two pebbles on every other vertex in the middle row, such that no three consecutive vertices have more than four pebbles. In either case, we need at most m pebbles.

Lower bound. We then show $\Pi_{OPT}(C_m \square P_3) \geq m$. By the Collapsing Lemma, we can collapse $C_m \square P_3$ into $C_m \square K_2$, which has an optimal pebbling number of m by *Theorem 2.3*. Since $\Pi_{OPT}(C_m \square P_3) \geq \Pi_{OPT}(C_m \square K_2)$, $\Pi_{OPT}(C_m \square P_3) \geq m$.

Theorem 6 *For* $m \geq 2$, $\Pi_{OPT}(P_m \square C_3) = \Pi_{OPT}(P_m \square P_3) = m + 1$.

Proof First, since $P_m \square P_3$ is contained in $P_m \square C_3$, $\Pi_{OPT}(P_m \square C_3) \leq \Pi_{OPT}(P_m \square P_3)$. Hence, if we can prove $\Pi_{OPT}(P_m \square P_3) \leq m+1$ and $\Pi_{OPT}(P_m \square C_3) \geq m+1$, we are done. We establish the upper and lower bounds separately.

Upper bound. First we show $\Pi_{OPT}(P_m \square P_3) \leq m+1$. If we put two pebbles on one vertex in the middle row and one pebble on all other vertices in the middle row, we will get a solvable distribution. Therefore, $\Pi_{OPT}(P_m \square P_3) \leq m+1$.

Lower bound. We now show $\Pi_{OPT}(P_m \square C_3) \geq m+1$. First let us collapse $P_m \square C_3$ into $P_m \square C_2$, where C_2 is a K_2 with a multi-edge. (See Figure 11.)

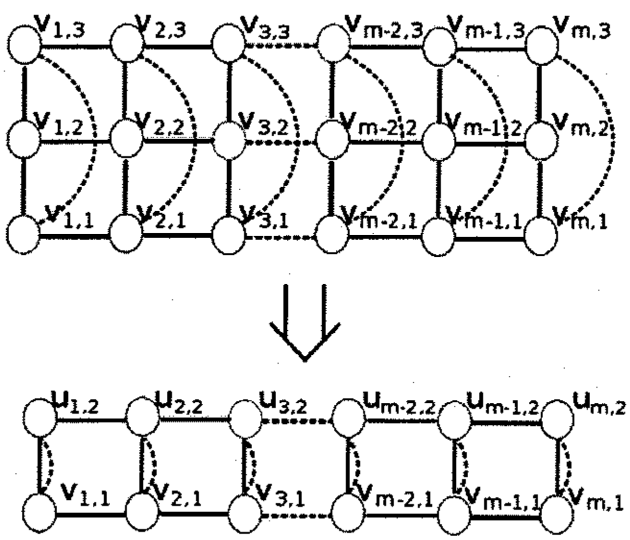

Fig. 11 Collapsing the top two rows of $P_m \square C_3$.

By the Collapsing Lemma, $\Pi_{OPT}(P_m \square C_3) \geq \Pi_{OPT}(P_m \square C_2)$. Since multi-edges do not affect pebbling numbers, $\Pi_{OPT}(P_m \square C_2) = \Pi_{OPT}(P_m \square K_2)$. Let us suppose for a contradiction that $\Pi_{OPT}(P_m \square C_3) = m$, then $m \geq \Pi_{OPT}(P_m \square C_2) = \Pi_{OPT}(P_m \square K_2)$. According to Theorem 4, this bound is tight, so the distribution on $P_m \square C_2$ can be decomposed into two types of fundamental blocks. (See Figure 8.)

When two adjacent vertices collapse into one vertex, the number of pebbles on the new vertex equals the sum of the number of pebbles on the original two vertices. Therefore, we can "un-collapse" $P_m \square C_2$ into $P_m \square C_3$ by transforming each vertex on the top row of $P_m \square C_2$ into a K_2 and distributing the pebbles on that vertex to the corresponding K_2. Without loss of generality, let us assume $u_{2,2}$ is occupied.

We discuss three possibilities in the following case analysis.

Case 1: Vertex $u_{2,2}$ has a single pebble and belongs to a 2-1 block. Under this configuration, $v_{1,1}, u_{1,2}, v_{3,1}$, and $u_{3,2}$ are unoccupied and $v_{2,1}$ has two pebbles. After un-collapsing, $v_{2,3}$ and $v_{2,2}$ will share one pebble. If the pebble remains on $v_{2,2}$, then $v_{1,3}$ is unreachable. If the pebble goes to $v_{2,3}$, then $v_{1,2}$ becomes unreachable. (Note that these might not be the only unreachable vertices.)

Case 2: Vertex $u_{2,2}$ has two pebbles and belongs to a 2-1 block. Under this configuration, $v_{1,1}, u_{1,2}, v_{3,1}$, and $u_{3,2}$ are unoccupied and $v_{2,1}$ has one pebble. After un-collapsing, $v_{2,3}$ and $v_{2,2}$ will share two pebbles. If both pebbles remain on $v_{2,2}$, $v_{1,3}$ is unreachable. If one pebble goes to $v_{2,3}$ and one stays on $v_{2,2}$, then $v_{1,3}$ is still unreachable. If both pebbles are placed on $v_{2,3}$, then $v_{1,2}$ becomes unreachable.

Case 3: Vertex $u_{2,2}$ has two pebbles and belongs to a 2-2 block. Under this configuration, $v_{1,1}, u_{1,2}, v_{2,1}, u_{3,2}, v_{4,1}$, and $u_{4,2}$ are all unoccupied and $v_{3,1}$ has two pebbles. After un-collapsing, $v_{2,3}$ and $v_{2,2}$ will share two pebbles. If both pebbles remain on $v_{2,2}$, $v_{1,3}$ is unreachable. If one pebble goes to $v_{2,3}$ and one stays on $v_{2,2}$, then again, $v_{1,3}$ is still unreachable.

If both pebbles are placed on $v_{2,3}$, then $v_{1,2}$ becomes unreachable.

Since the un-collapsing of $P_m \Box C_2$ into a $P_m \Box C_3$ makes an optimal distribution no longer solvable, there exists no solvable distribution with only m pebbles on $P_m \Box C_3$, which means $\Pi_{OPT}(P_m \Box C_3) \geq m + 1$. This completes our proof.

5 Introduction to Fractional Pebbling

In this section we give background relating to *fractional pebbling*, introduced in [10] and extended in [5]. We begin with basic definitions from [10]. For a graph G, a function $D : V \to \mathbb{R}^{\geq 0}$ is called a *continuous distribution* on G. As in an integer-valued distribution, the *size* of D is given by $|D| = \sum_{v \in V} D(v)$. A *continuous pebbling move* from a vertex v to an adjacent vertex u takes away t ($t \in \mathbb{R}, t \geq 0$) pebbles at v and adds only $\frac{t}{2}$ pebbles to u. The *optimal fractional pebbling number* of G, denoted $\hat{\pi}_{OPT}(G)$, is the least t ($t \in \mathbb{R}^+$) such that some particular distribution of t pebbles is solvable.

Lemma 8 gives a criterion to determine whether a vertex v is reachable or not.

Lemma 8 *[5] Let D be a distribution on a graph G. Then there is a sequence of fractional pebbling moves starting from D which places a pebble on $r \in V$ if and only if $\sum_{v \in V} D(v) 2^{-dist(v,r)} \geq 1$.*

Lemma 9 relates directly to one of our graphs of interest, the infinite grid G_{inf}, which is also a vertex-transitive graph.

Lemma 9 *[5] If $G = (V, E)$ is a vertex-transitive graph, then the function $f : V \to \mathbb{R}^+$ given by $f(u) = \sum_{v \in V} 2^{-dist(v,u)}$ is constant for all $u \in V$.*

Theorem 7 *[5] If G is a vertex-transitive graph with n vertices, an optimal continuous distribution on G is obtained by putting $\frac{1}{m}$ pebbles on each vertex in G, where m is the constant $\sum_{v \in V} 2^{-dist(v,u)}$. Therefore, $\hat{\pi}_{OPT}(G) = \frac{n}{m}$.*

Since G_{inf} has infinite vertices, it does not have an optimal fractional pebbling number, but we can still find an optimal (or better) distribution by comparing the covering ratio between different distributions.

A distribution D on a graph G is said to be *optimal* if the covering ratio of D is no less than any other distributions on G.

In the next section, we focus on the graph G_{inf} which does not have a finite optimal pebbling number, but we can still find an optimal distribution on G_{inf} by figuring out what the greatest covering ratio of a distribution D on G_{inf} is.

6 New Results about G_{inf}

Lemma 10 *On G_{inf}, the shape of a unit is a square and a unit with $2^n + k$ ($n \in \mathbb{N} \cup \{0\}, 0 \le k < 2^n$) pebbles covers $(n+1)^2 + n^2$ vertices. (See Fig.12)*

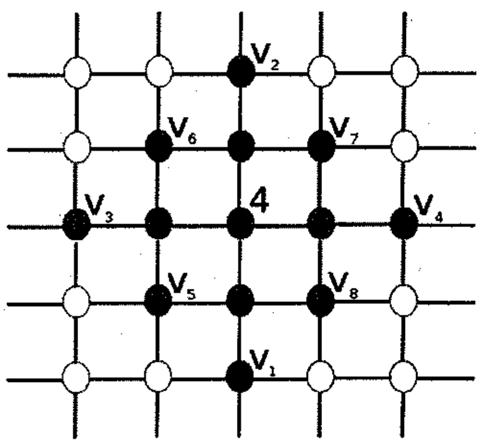

Fig. 12 A unit with four pebbles on G_{inf}.

Proof The source is equidistant from all the boundary vertices, so the shape looks like a square. If the source of the unit has $2^n + k$ pebbles, the maximum distance it can travel is n. Therefore, the total number of vertices covered is (counting from inside out) $1 + 4 + 8 + ... + 4n = 1 + 4(1 + 2 + ... + n) = 1 + 4n(n+1)/2 = 2n^2 + 2n + 1 = n^2 + (n+1)^2$.

In addition, a unit has four *corner vertices* (vertices that are adjacent to three unreachable vertices). We call the rest of the boundary vertices *non-corner vertices.* For example, in Figure 12, vertices v_1, v_2, v_3, and v_4 are corner vertices and vertices v_5, v_6, v_7, and v_8 are non-corner vertices. Unlike a block on $P_m \square K_2$, which has only two types of ends, a block on G_{inf} can take many shapes, but whatever the shape is, its border must contain a certain configuration.

Lemma 11 *On G_{inf}, if a block has more than one pebble, then a boundary vertex of this block is adjacent to at least one 2-reachable or 3-reachable vertex. Therefore, the border of the block must possess the following configuration. (Note that this configuration might not look exactly the same as the border of a block, because the other vertices (striped ones) might also be covered.) (See Figure 13.)*

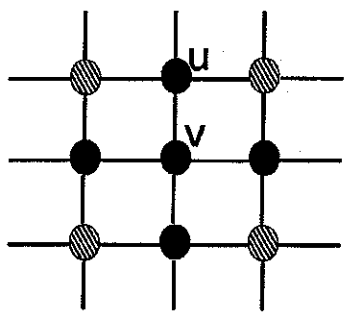

Fig. 13 The minimal configuration on the border of a block.

Proof Without loss of generality, let us assume that u is a boundary vertex of a block. Therefore, it must be exactly 1-reachable. If u is occupied, it has one pebble. But since u is not 2-reachable, no pebble can be sent from its adjacent vertices to u, which means u itself is a block and has only one pebble. This contradicts our assumption.

Hence, u is not occupied. This requires that some vertex v adjacent to u is able to send one and only one pebble to u. Therefore, v is either 2-reachable or 3-reachable. Then all vertices adjacent to v must also be covered and thus we have the minimal configuration shown in Figure 13.

In the next lemma, we use the definition of *covering ratio* in section 3 again, which is given by

$$\frac{Cov(D)}{|D|}.$$

Lemma 12 *The covering ratio of a single unit on G_{inf} is at most 3.25.*

Proof According to Lemma 10, a unit with $2^n + k$ ($n \in \mathbb{N} \cup \{0\}, 0 \leq k < 2^n$) pebbles covers $(n+1)^2 + n^2$ vertices. The covering ratio is given by $\frac{(n+1)^2+n^2}{2^n+k}$. To maximize this value, set $k = 0$. After using the derivative tests, we find that this value reaches its unique maximum when $n \approx 2.296$. However, $n \in \mathbb{N} \cup \{0\}$, so $n = 2$ or $n = 3$. When $n = 2$, the covering ratio is 3.25, and when $n = 3$, the covering ratio is 3.125. Hence, the maximum covering ratio is 3.25.

Lemma 13 *The marginal covering ratio of a unit on G_{inf} is at most 4.25 if the interaction happens only on the boundary vertices (1-reachable vertices).*

Proof Let D be an initial distribution and U be a unit that interacts with D. Assume that the source of U has $2^n + k(n \in \mathbb{N}, 0 \leq k < 2^n)$ pebbles. If a unit does not interact with others, we say this unit is *lonely*. We assume that D does not contain lonely units with one pebble because if D contains those units, we can remove them first and add them after U has been added. Since the interaction only happens on boundary vertices, these units will yield a marginal covering ratio of 3.

Case 1: $n = 0$. The unit has only one pebble and it falls on a 1-reachable vertex. According to Lemma 11, this 1-reachable vertex is already adjacent to at least one covered vertex, so at most three new vertices can be covered with this added source. The marginal covering ratio is 3. This completes *Case 1*.

Since the interaction happens only on the boundary vertices, these vertices become 2-reachable after the interaction. Furthermore, if the source of the unit has more than one pebble, then two pebbles must come to an interaction vertex from two different edges, which means at least two of the vertices adjacent to an interaction vertex are already covered. Now since each vertex is of degree 4, at most two new vertices can be covered by this interaction.

(Note that only when the interaction happens on the corner vertices can it cover two new vertices. If it happens on the non-corner vertices, one new vertex is covered but the size of the coverage remains unchanged.)

Case 2: $n = 1$. The unit has two or three pebbles. If all corner vertices are interaction vertices, there should be eight new vertices covered. However, there are overlaps because the unit is too small. Hence, only six new vertices can be covered, and the maximal marginal coverage is 7 (including the source and the six new covered vertices). The marginal covering ratio is $\frac{7}{2} = 3.5$ or $\frac{7}{3} \approx 2.33$. We take the larger one, which is 3.5.

Case 3: $n \geq 2$. The unit now has more than three pebbles, which means the corner vertices are far enough such that they will not interfere with each other. Under this condition, each interaction on the corner vertex increases the size of the coverage by 1, and since their are four corners, the size of the coverage increases by at most 4. Hence the maximal marginal coverage is $(n+1)^2+n^2+4$ and the maximal marginal covering ratio is $\frac{(n+1)^2+n^2+4}{2^n}$. Using the derivative tests, we see that the function is monotonically decreasing. Therefore, the maximum value is obtained when $n = 2$. The maximal marginal covering ratio is 4.25 ($n = 2, k = 0$).

Combining all three cases above, we see that 4.25 is an upper bound for the marginal covering ratio of a unit if it interacts only on the boundary vertices.

Definition 8 Let D be a continuous distribution on a graph G. We define the *weight* of a vertex r under D, written as $W_D(r)$, to be $\sum_{v \in V} D(v)2^{-dist(v,r)}$.

Therefore, Lemma 8 can be written in the following way: Given a distribution D on a graph G, a vertex v is reachable if and only if $W(r) \geq 1$.

Definition 9 Let D be a continuous distribution on a graph G. We define the *excess weight* of a vertex v under D, denoted $\hat{W}_D(v)$, to be $W_D(v) - 1$ if $W_D(v) > 1$ and 0 if $W_D(v) \leq 1$.

Excess weight describes the amount of weights that do not contribute to the size of the coverage because a vertex needs only a weight of 1 to be reachable.

Definition 10 Given a continuous distribution D on a graph G, the *covering ratio ceiling* is given by

$$\frac{\sum_{v \in V} W_D(v) - \sum_{v \in V} \hat{W}_D(v)}{\sum_{v \in V} D(v)}.$$

The covering ratio ceiling is also the highest possible covering ratio for a configuration.

Lemma 14 *The covering ratio ceiling of a unit on G_{inf} is at most 9 and it decreases monotonically as the number of pebbles on the unit grows.*

Proof If we put one single pebble on a source r, $\sum_{v \in V} W(v)$ is given by $1+4*2^{-1}+8*2^{-2}+12*2^{-3}+... = 1+\sum_{i=1}^{\infty} 4i*2^{-i} = 1+4\sum_{i=1}^{\infty} i*2^{-i} = 1+4*2 = 9$. If $W(v) > 1$ for some vertex v, v is covered, but the portion of weight that is above 1 cannot be transferred to another vertex. Therefore, they should not be counted when we calculate the covering ratio. Therefore, if we consider pebbles on just one vertex, we give a table of excess weights and covering ratio ceiling below.

Table 1 Excess Weights and Covering Ratio

Num. of Pebbles on Source	Total Weight	Excess Weight	Covering Ratio Ceiling
1	9	0	9.00
2	18	1	8.50
3	27	4	7.67
4	36	7	7.25
5	45	12	6.60
6	54	17	6.17
7	63	22	5.86
8	72	27	5.63
...	...	...	...

As we can see, the covering ratio ceiling decreases as the number of pebbles on the source grows because more vertices will have excess weights with the increase in the number of pebbles.

Definition 11 Suppose we have a continuous distribution D on a graph G. Now if we put some extra pebbles on G to obtain a new distribution D', then the *marginal covering ratio ceiling* of these pebbles is given by

$$\frac{(\sum_{v\in V} W_{D'}(v) - \sum_{v\in V} \hat{W}_{D'}(v)) - (\sum_{v\in V} W_D(v) - \sum_{v\in V} \hat{W}_D(v))}{\sum_{v\in V} D'(v) - \sum_{v\in V} D(v)}.$$

Lemma 15 *The marginal covering ratio ceiling of a unit U on G_{inf} is no more than 6 if the unit interacts not only on the boundary vertices.*

Proof Suppose we have a distribution D that excludes lonely units with one pebble. Now we add a unit U onto the distribution. (If D contains lonely units with one pebble, remove them first and add them after U has been added. According to the Table 2, such units have a maximal marginal covering ratio ceiling of 6.)

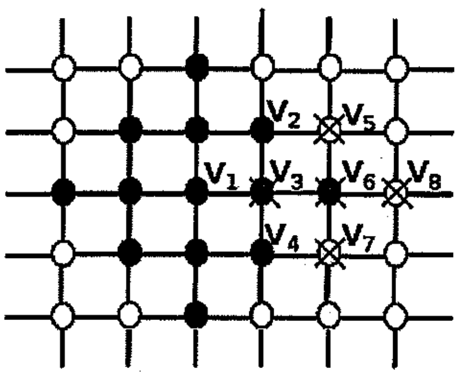

Fig. 14 Interaction between a unit and a distribution.

If U has more than one pebble, there are at least two interaction vertices as shown in Figure 14 (the shaded vertices belong to U while the crossed ones belong to D). Note that D can have more vertices covered but the crossed ones are its minimal configuration by Lemma 11. Note also that Figure 14 is not the only way to have an interaction but it gives a configuration that yields the least excess weight. Since v_3, v_5, v_6, v_7, and v_8 are already covered by D, any weight added on them will become excess weight. Because v_6 is 1-reachable from U, it receives a weight of 1 from U, which should be counted as excess weight. Similarly, v_5, v_7, v_8 each contribute an excess weight of at least 0.5. Besides, v_1, v_2, v_3, and v_4 initially do not have excess weights on them but after the addition of U, their weights acquired from D become excess weights. Since v_1, v_2, v_4 each yield an excess weight of at least 0.5 and v_3 yields at least 1, the total amount of excess weight increases by $1+0.5*3+0.5*3+1 = 5$. (Actually, the larger the unit, the more the excess weight, so 5 is but a conservative value.)

If U has only one pebble on its source, its adjacent vertices must be covered by D because we assume the interaction does not happen on boundary vertices. Under this circumstance, the excess weight increases by at least $1 + .5 * 4 = 3$.

Table 2 Marginal Weights and Marginal Covering Ratio

# Pebbles	Total Weight	Min. Excess Weight	Max. Marg. Cov. Ratio Ceiling
1	9	3	6.0
2	18	6	6.0
3	27	9	6.0
4	36	12	6.0
5	45	17	5.6
6	54	22	5.3
7	63	27	5.1
8	72	32	5.0
...	...	...	...

Therefore, we give a table of marginal covering ratio ceiling above. The total weight always equals to 9 times the number of pebbles, so in order to maximize marginal covering ratio ceiling, we minimize the increase in excess weight.
As we can see from the table, the marginal covering ratio ceiling is at most 6.

Theorem 8 *The covering ratio of any distribution D on G_{inf} is at most 6.*

Proof We prove this by induction.
Base Case: The covering ratio of a single unit is less than 6 according to Lemma 12.
Inductive Hypothesis: For $n \geq 2$, suppose the covering ratio of a distribution that consists of n units is no more than 6. We want to show that a block that consists of $n + 1$ units also has a covering ratio of 6 or less.
Analysis: Since the marginal covering ratio of a unit is at most 6 by Lemma 13 and Lemma 15, we cannot obtain a distribution whose covering ratio is greater than 6.

Conjecture 1 The covering ratio of any distribution D on G_{inf} is at most 3.75.

We can consider all vertices in G_{inf} as lattice points on a plane. Therefore, they can be expressed as (x, y), where $x, y \in \mathbb{Z}$. A distribution that has a covering ratio of 3.75 has the following property:

- A vertex has either four pebbles or zero pebbles.
- If (x, y) has four pebbles, then $(x + 4, y)$, $(x - 4, y)$, $(x, y + 4)$, $(x, y - 4)$ all have four pebbles on them.
- One vertex has four pebbles.

Figure 15 gives a local distribution of covering ratio 3.75. The shaded vertices are vertices with four pebbles and the crossed one is the unreachable vertex for this distribution.

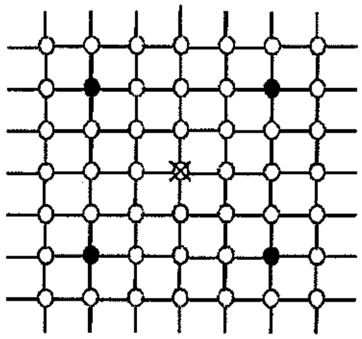

Fig. 15 A sample distribution of covering ratio 3.75.

Conjecture 2 If $\forall x, y \in \mathbb{Z}$, (x, y) is reachable under a distribution D on G_{inf}, then the covering ratio of D is at most 3.25.

For example, Figure 16 uses infinite lonely sources with four pebbles to make the entire graph reachable (the filled vertices are the ones with four pebbles and the squares indicate their coverage) and the covering ratio of such a distribution is 3.25 as proved in Lemma 12.

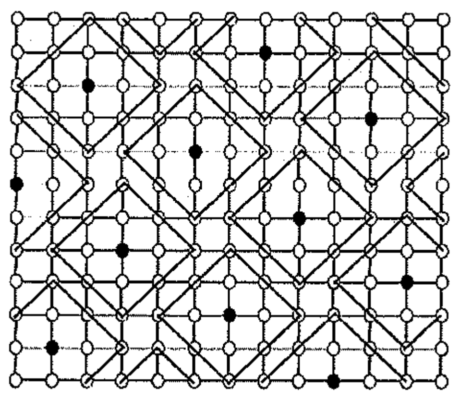

Fig. 16 A sample distribution of covering ratio 3.25 that packs G_{inf}.

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