Domination Cover Pebbling: Structural Results

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Abstract

This paper continues the results of "Domination Cover Pebbling: Graph Families." An almost sharp bound for the domination cover pebbling (DCP) number, $\psi(G)$, for graphs G with specified diameter has been computed. For graphs of diameter two, a bound for the ratio between $\lambda(G)$, the cover pebbling number of G, and $\psi(G)$ has been computed. A variant of domination cover pebbling, called subversion DCP is introduced, and preliminary results are discussed.

1 Introduction

Given a graph G we distribute a finite number of indistinguishable markers called *pebbles* on its vertices. Such an arrangement of pebbles, which can also be thought of as a function from $V(G)$ to $\mathbb{N} \cup \{0\}$, is called a *configuration*. A *pebbling move* on a graph is defined as removing two pebbles from one vertex, throwing one away, and moving the other to an adjacent vertex. Most research in pebbling has focused on a quantity known as the *pebbling number* $\pi(G)$ of a graph, introduced by F. Chung in [2], which is defined to be the smallest integer n such that for every configuration of n pebbles on the graph and for any vertex $v \in G$, there exists a sequence of pebbling moves starting at this configuration and ending in a configuration in which there is at least one pebble on v. A new variant of this concept, introduced in by Crull et al. in [6], is the *cover pebbling number* $\lambda(G)$, defined as the minimum number m such that for any initial configuration of at least m pebbles on G it is possible to make a sequence of pebbling moves after which there is at least one pebble on every vertex of G.

In a recent paper [7] the authors, along with Gardner, Godbole, Teguia, and Vuong, have introduced a concept called domination cover pebbling and presented some preliminary results. Given a graph G, and a configuration c, we call a vertex $v \in G$ *dominated* if a pebble covers v or v is adjacent to a vertex with a pebble on it. We call a configuration c' *domination cover pebbling solvable*, or simply *solvable*, if there is a sequence of pebbling moves starting at c' after which every vertex of G is dominated. We define the *domination cover pebbling number* $\psi(G)$ to be the minimum number n such that any initial configuration of n pebbles on G is domination cover pebbling solvable.

The set of covered vertices, S, in the final configuration depends, in general, on the initial configuration—in particular, S need not equal a minimum dominating set. For instance, consider the configurations of pebbles on P_4, the path on four vertices, as shown in Figure 1:

Figure 1: An example where two different initial configurations produce two different domination cover solutions.

For the graph on the left, we make pebbling moves so that the first and third vertices (from left to right) form the vertices of the dominating set. However, for the graph on the right, we make pebbling moves so that the second and fourth vertices are selected to be the vertices of the dominating set. In some cases, moreover, it takes more vertices than are in the minimum dominating set of vertices to form the domination cover solution. For example, in Figure 2 consider the case of the binary tree with height two, where the minimum dominating set has two vertices, but the minimal dominating set possible for a domination cover solution has three vertices. This corresponds to several possible starting configurations, for example the configuration pictured, the configuration with a pebble at the leftmost bottom vertex and four pebbles at the root, and the configuration

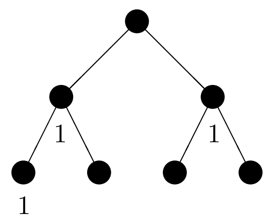

Figure 2: A reachable minimal configuration of pebbles on B_2 that forces a domination cover solution.

with one and ten pebbles at the leftmost and rightmost bottom level vertices respectively.

The above two facts constitute the main reason why domination cover pebbling is nontrivial. We refer the reader to [8] for additional exposition on domination in graphs, and to [7] for some further explanation of the domination cover pebbling number, including the computation of the domination cover pebbling number for some families of graphs.

One way to understand the size of the numbers $\pi(G)$, $\lambda(G)$, and $\psi(G)$ is to find a bound for the size of these numbers given the diameter of G and the number of vertices. This has been done for $\pi(G)$ for graphs of diameter two in [5] and for graphs of diameter three in [1]. Recently, an asymptotically tight bound for graphs of diameter four was found in [9]. These authors also improved the asymptotic bound of [1]. A theorem proven in [10] and [11] gives as a corollary a sharp bound for graphs of all diameters, which was originally established by other means in [12]. In this paper, we prove that for graphs of diameter two with n vertices, $\psi(G) \leq n-1$. For graphs of diameter d, we show $\psi(G) \leq 2^{d-2}(n-2)+1$. We also compute that the ratio $\lambda(G)/\psi(G) \geq 3$ for graphs of diameter two.

Another way to extend cover pebbling is called *subversion domination cover pebbling*. A parameter ω used in calculating the vertex neighbor integrity of a graph G counts the size of the largest undominated connected subset of G. When $\omega = 0$, this corresponds to domination cover pebbling. To conclude this paper, we provide some preliminary results for this generalized parameter.

2 Diameter Two Graphs

In the next few sections, we will present structural domination cover pebbling results.

Theorem 2.1. *For all graphs G of order n with maximum diameter two, $\psi(G) \le n-1$.*

Proof. First, we show this bound is sharp by exhibiting a graph G such that $\psi(G) > n-2$. Consider the star graph on n vertices, and place a pebble on all of the outer vertices except one. This configuration of pebbles does not dominate the last outer vertex. Hence, $\psi(G) > n-2$.

To prove the theorem, we will show that, given a graph G of diameter two on n vertices, any configuration c of $n-1$ pebbles on G is solvable.

Given such a graph configuration c, let S_1 be the set of vertices $v \in V(G)$ such that $c(v) > 1$. Let S_2 be the set vertices $w \in V(G)$ such that $c(w) = 0$ and w is adjacent to some vertex of S_1, and let S_3 be the rest of the vertices, the ones that are neither in S_1 nor adjacent to a vertex of S_1. Let $a := |S_2|$, and $b := |S_3|$. Given an arbitrary configuration c, define the *pairing number* $P(c)$ to be $\sum_{v\in G} \max\{0, \frac{c(v)-1}{2}\}$. It can easily be checked that $P(c) = \frac{a+b-1}{2}$. Note that if $P(c) = k$ then c contains at least $\lceil k \rceil$ pairs of pebbles on vertices, which means that we can make at least $\lceil k \rceil$ pebbling moves. Also, note that every vertex in G is at distance at most two from some vertex in S_1. This ensures that every vertex in S_3 is adjacent to a vertex in S_2. Also, if some vertex in S_1 is not adjacent to a vertex of S_2, it must be adjacent only to vertices in S_1. Since this vertex has distance at most two from any other vertex on the graph, we conclude that every vertex of the graph is either in S_1 or adjacent to a vertex of S_1, meaning the G is already dominated by covered vertices, as desired. Therefore, it suffices to consider the case in which S_2 is a dominating set of G.

First, suppose that $a \le b$. In this case, $P(c) \ge \frac{2a-1}{2}$. Hence, there are at least a disjoint pairs of pebbles that can be moved from elements in S_1 to S_2. For each uncovered vertex $v \in S_2$, if possible, move a pair of pebbles from an adjacent element of S_1 to put a pebble on v. After this is done for as many vertices of S_2 as possible, let L be the set vertices in S_2 which are still uncovered. Note that these vertices are necessarily at distance two from all remaining pairs of pebbles. Furthermore, since S_1 initially had at least a disjoint pairs of pebbles, there remain at least as many pairs as there are vertices in L. If this number is zero, the dominating set S_2 is covered and we are done. Otherwise, we nonetheless now know S_3 is dominated because if there were some vertex y that were adjacent to only those elements of S_2, which are also in L, then the minimum distance between y and a vertex in S_1 with a pair of pebbles is three, which is impossible. However, it may be the case for some $z \in L$ that all vertices in S_1 that z was adjacent to lost their pebbles, and if this is the case, move a pair of pebbles from S_1 so that z is dominated (this always possible since our graph has diameter two). With the $|L|$ pairs we of pebbles we have, we can ensure each vertex of L is dominated. After this is done, G will be completely dominated by covered vertices.

Now consider the case $a > b$. Given any vertex v in S_3 and a pair of pebbles on a vertex $w \in S_1$, we can use this pair to move to a vertex between v and w, which is clearly in S_2. In order to construct a domination cover solution, do this whenever necessary for each vertex of S_3, but only using those pairs of pebbles which can be removed from vertices having at least three pebbles. Further, since $a > b$, there is some $u \in S_2$ that is adjacent to at least two vertices in S_3. Choose this u first. After this process, all the remaining vertices with multiple pebbles on them contain two pebbles. If every vertex either contains a pebble or is adjacent to a vertex with a pebble, we are finished and $\psi(G) \leq n - 1$. Let m_1 be the number of moves that have been made from vertices containing at least three pebbles. Then we know that m_1 vertices in S_2 now have pebbles on them. Furthermore, there are at least $m_1 + 1 \leq b$, vertices are dominated in S_3. The additional vertex in S_3 comes from our choice of u for the first pebbling move. If initially there were no vertices with at leas three pebbles, the first pebbling move from a vertex with only a pair of pebbles will be chosen to dominate two vertices in S_3.

Otherwise there are still vertices in S_3 that are undominated. To handle this consider the following process. First, if there is some unpebbled vertex in S_2, say u_1, that is adjacent to only one vertex in S_1, say v_1, with multiple pebbles and is not adjacent to any other vertex with pebbles, then "lock" u_1 and v_1 together so that the pebbles in v_1 can not be moved unless u_1 is dominated elsewhere. Notice that multiple vertices in S_2 may be locked with the same v_1, and that this only helps our bound. Notice that for each locking, the available pairing number decreases by 1/2 but the effective size of a, the set of vertices in S_2, decreases by one. Call this new set $\hat{S}_2$, where $|\hat{S}_2| = a'$. Second, if there is still an undominated vertex (in S_3), make a pebbling move with an unlocked vertex. After this step, check the vertices in S_2 to see whether more vertices in S_1 need to be locked or unlocked. Repeat the first and second steps until the graph is dominated.

It remains to show this process terminates and that we have enough pebbles to perform it. In this process, once a vertex is dominated, it remains dominated. This is clear for vertices in S_1 and S_3, and for S_2, the locking procedure ensures that any dominated vertex does not become undominated. Thus, this process terminates. To see that we have enough pairs of pebbles, observe that each pebbling move with a vertex of at least three or more pebbles covers some vertex in S_2 and dominates at least one vertex in S_3. The first pebbling move made dominates at least two vertices in S_3. As a result, the remaining number of vertices to dominate is at most $a + b - 2m_1 - 1$. If the process terminates at this point, then our configuration had enough pebbles to dominate the graph. Once all vertices with three or more pebbles have been eliminated (and possibly one pebbling move with a vertex having two pebbles if it is the first move), the remaining pairing number is $\frac{a+b-2m_1-1}{2}$. At this point, there are at most $a+b-2m_1-1$ vertices that are unpebbled if in S_2 and undominated if in S_3. Each pebbling move in the second paragraph of our process dominates at least one new vertex in S_3 and the pairing number decreases by 1/2. Each locked pair of vertices in S_1 and S_2 dominates two vertices and the pairing number decreases by 1/2. As a result, we have a sufficiently large pairing number so that our algorithm can be completed. This finishes the proof.

□

We can apply this theorem to prove a result about the ratio between the cover pebbling number and the domination cover pebbling number of a graph. We conjecture that this ratio holds for all graphs, but it does not seem that this can be directly proven using the structural bounds in this paper.

Theorem 2.2. *For all graphs G of order n with diameter two, $\lambda(G)/\psi(G) \geq 3$.*

Proof. First, suppose that the minimum degree of a vertex of G is less than or equal to $\lceil \frac{n-1}{2} \rceil$. By the previous theorem, we know that the maximum value of $\psi(G)$ is $n-1$. We now construct a configuration of pebbles on G such that $\lambda(G) \geq 3n-3$. Place $3n-3$ pebbles on any vertex v that has a degree less than $\lceil \frac{n-1}{2} \rceil$. It takes 2 pebbles to cover solve each vertex adjacent to v, at most $\lceil \frac{n-1}{2} \rceil$, and all the remaining vertices require four pebbles. Since there are at least as many vertices a distance of two away from v as there are distance one away from v, $3n-3$ pebbles or more are required to cover pebble all of the vertices except for v. Thus for this class of graphs, $\lambda(G) \geq 3n-3 \geq 3\psi(G)$.

Now suppose that the minimum degree k of a vertex in G is greater than $\lceil \frac{n-1}{2} \rceil$. By a similar argument as the previous paragraph, notice that $\lambda(G)$ for any diameter two graph is at least $4n-2m-3$, where m is the minimum degree of a vertex of G. Since $\lambda(G) \geq 4n-2m-3$, it suffices to show we can always solve a configuration c of $\lfloor \frac{4n-2m-3}{3} \rfloor = k$ pebbles on G. Given a particular value for m between $\lceil \frac{n+1}{2} \rceil$ and $n-1$, we will construct a domination cover solution.

As long as there exist vertices of G that have at least three pebbles and are adjacent to an unoccupied vertex, we haphazardly make moves from such vertices to adjacent unoccupied vertices. We claim that the resulting configuration has the desired property that the set of occupied vertices are a dominating set of G. First suppose that the algorithm is forced to terminate while there remains some vertex v having at least three pebbles. Then this vertex must be adjacent only to occupied vertices of G, and since the diameter of G is two, these neighbors of v form a dominating set of G. Otherwise, if every vertex has less than three pebbles, it can easily be checked that the number of occupied vertices is now $\sum_{v\in G}\lceil\frac{c(v)}{2}\rceil \geq \lceil\frac{k}{2}\rceil$. Since the minimum degree of a vertex in G is m, by the pigeonhole principle, if we now have $n-m$ or more vertices covered by a pebble, then every vertex of G is dominated. So if $\lceil\frac{k}{2}\rceil \geq n-m$, we are finished. We see that

$$\left\lceil \frac{\lfloor\frac{4n-2m-3}{3}\rfloor}{2} \right\rceil \geq \left\lceil \frac{\frac{4n-2m-5}{3}}{2} \right\rceil = \left\lceil \frac{4n}{6} - \frac{m}{3} - \frac{5}{6} \right\rceil$$

Therefore, we are done if

$$\left\lceil \frac{4n}{6} - \frac{m}{3} - \frac{5}{6} \right\rceil \geq n-m,$$

which is equivalent to

$$n \leq \left\lceil \frac{4n}{6} + \frac{2m}{3} - \frac{5}{6} \right\rceil.$$

This inequality holds for $m \geq \lceil\frac{n+1}{2}\rceil$. Therefore, we have completed this case and have shown that for all graphs G of diameter two, $\lambda(G)/\psi(G) \geq 3$. □

We now prove a more general bound for graphs of diameter d.

3 Graphs of Diameter d

Theorem 3.1. *Let G be a graph of diameter $d \geq 3$ and order n. Then $\psi(G) \leq 2^{d-2}(n-2)+1$.*

Throughout the proof, we adopt the convention that if G is a graph and V and W are subsets of $V(G)$ and $v \in V(G)$ then $d(v, W) = \min_{w\in W} d(v, w)$ and $d(V, W) = \min_{v\in V} d(v, W)$. Also, for any set $S \subseteq V(G)$ we of course let $S^C = V(G) \setminus S$.

Proof. First, we define the *clumping number* χ of a configuration c' by

$$\chi(c') := \sum_{v\in G} 2^{d-2} \max\left(\left\lfloor \frac{c'(v)-1}{2^{d-2}} \right\rfloor,\ 0\right).$$

The clumping number counts the number of pebbles in a configuration which are part of disjoint "clumps" of size 2^{d-2} on a single vertex, with one pebble on each occupied vertex ignored.

Now let c be a configuration on G of size at least $2^{d-2}(n-2)+1$. We will show that c is solvable by giving a recursively defined algorithm for solving c through a sequence of pebbling moves. First, we make some definitions to begin the algorithm:

- $c_0 = c$.
- $A_0 = \{v \in G \ : \ c(v) > 0\}$.
- $B_0 = \{v \in G \ : \ c(v) \geq 2^{d-2} + 1\}$.
- $C_0 = V(G) - A_0$.
- $D_0 = \emptyset$.

We will describe our algorithm by recursively defining a sequence of configurations c_p and four sequences A_p, B_p, C_p, and D_p of sets of vertices. At each step, we will need to make sure a few conditions hold, to ensure that the next step of the algorithm may be performed. For each m, we will insist that:

1. For every $v \in C_m \cup D_m$, $c_m(v) = 0$ and for every $v \in A_m$, $c_m(v) > 0$.
2. $\chi(c_m) \geq 2^{d-2}(|C_m| - 1)$.
3. $|C_m| \leq |C_0| - m$.
4. $B_m = \{v \in G \ : \ c_m(v) \geq 2^{d-2} + 1\}$.
5. If both $B_m \neq \emptyset$ and $D_m \neq \emptyset$, $d(B_m, D_m) = d$; if $D_m \neq \emptyset$, there always exists some $v \in G$ such that $d(v, D_m) = d$, even if $B_m = \emptyset$.
6. A_m, C_m, and D_m are pairwise disjoint and $A_m \cup C_m \cup D_m = V(G)$.
7. Every vertex of D_m is dominated by c_m.
8. There exists a sequence of pebbling moves transforming c to c_m.

Note by 1, 4, and 6, we will always have $B_m \subseteq A_m$. Also, by 1, 6, and 7, every vertex of G which is not dominated by c_m is in C_m.

For $m = 0$, only condition 2 is not immediately clear. To verify it, note that

$$\begin{aligned}
\chi(c) &= \sum_{v \in G} 2^{d-2} \max\left(\left\lfloor \frac{c(v) - 1}{2^{d-2}} \right\rfloor, \ 0 \right) \\
&= \sum_{v \in A_0} 2^{d-2} \left\lfloor \frac{c(v) - 1}{2^{d-2}} \right\rfloor \\
&\geq \sum_{v \in A_0} 2^{d-2} \left(\frac{c(v)}{2^{d-2}} - 1 \right).
\end{aligned}$$

Using the fact that the size of c is at least $2^{d-2}(n-2)+1$, and $|C_0| = n-|A_0|$, we see

$$\chi(c) \geq (2^{d-2}(n-2)+1) - 2^{d-2}|A_0| = 2^{d-2}(|C_0|-2)+1.$$

From the definition of χ, it is apparent that $2^{d-2} | \chi(c)$. Thus, we indeed must have

$$\chi(c) = \chi(c_0) \geq 2^{d-2}(|C_0|-1).$$

Suppose for some $p-1 > 0$ we have defined $c_{p-1}, A_{p-1}, B_{p-1}, C_{p-1}$, and D_{p-1} and the above conditions hold when $m = p-1$. We shall assume that there is some vertex in C_{p-1} which is not dominated by c_{p-1}, for otherwise, by conditions 6, 7 and 8, c is solvable and we are done. Thus $|C_{p-1}| \geq 1$. But suppose $|C_{p-1}| = 1$. Call this single vertex v. Since it is non-dominated, it is adjacent to only uncovered vertices. These vertices cannot be in C_{p-1} for $|C_{p-1}| = 1$, and they are not in A_{p-1}, because every vertex in A_{p-1} is covered by property 1. So every vertex adjacent to v is in D_{p-1}. Invoke property 5 to choose a $w \in G$ for which $d(w, D_{p-1}) = d$. Any path from w to v passes through one of the vertices in D_{p-1} which is adjacent to v, and is thus of length at least $d+1$, so $d(w,v) \geq d+1$, contradicting the assumption that G has diameter d. We have now shown that, if C_{p-1} has a non-dominated vertex, then $|C_{p-1}| \geq 2$. In this case, we will have $\chi(c_{p-1}) \geq 2^{d-2}$, ensuring the existence of some clump of size 2^{d-2}, and thus that B_{p-1} is non-empty. Therefore, we will always implicitly assume that $B_{p-1} \neq \emptyset$.

Case 1: $d(B_{p-1}, C_{p-1}) \leq d-2$

In this case, we choose $v' \in B_{p-1}$ and $w' \in C_{p-1}$ for which $d(v', w') \leq d-2$ and move $2^{d(v',w')}$ pebbles from v' to w', leaving one pebble on w' and at least one on v'. We let c_p be the configuration of pebbles resulting from this move. Let $C_p = C_{p-1} \setminus w'$. Thus $|C_p| = |C_{p-1}| - 1 \leq |C_0| - (p-1) - 1$ and we see that condition 3 holds when $m = p$. Furthermore, we have used at most one clump of 2^{d-2} pebbles so

$$\chi(c_p) \geq \chi(c_{p-1}) - 2^{d-2} \geq 2^{d-2}(|C_{p-1}|-1) - 2^{d-2} = 2^{d-2}(|C_p|-1)$$

and therefore condition 2 holds for p. Also, we let $A_p = A_{p-1} \cup \{w'\}$, let $C_p = C_{p-1}w'$, and $D_p = D_{p-1}$ (now, clearly condition 6 holds.) We again let $B_p = \{v \in G \ : \ c_p(v) \geq 2^{d-2}+1\}$, which simply means that we have possibly removed v' from B_{p-1} if v' now has less than $2^{d-2}+1$ pebbles. Thus $B_p \subseteq B_{p-1}$, and now 1, 4, 5, 7, and, 8 are all easily seen to hold for $m = p$.

Case 2: $d(B_{p-1}, C_{p-1}) \geq d-1$.

If every vertex in C_{p-1} is dominated by A_{p-1}, we are done. Otherwise, let w' be some non-dominated vertex in C_{p-1}. Clearly, w' is at distance $d-1$ or d from B_{p-1}. Suppose $d(B_{p-1}, w') = d-1$. Then w' is adjacent to some (non-covered) vertex w'' at distance $d-2$ from B_{p-1}. By condition 1, every vertex of G which is not covered by c_{p-1} is in $C_{p-1} \cup D_{p-1}$. But $d(B_{p-1}, C_{p-1}) \geq d-1$ and by 5, $d(B_{p-1}, D_{p-1}) = d$ so $w'' \notin C_{p-1} \cup D_{p-1}$. This contradiction means that $d(w', B_{p-1}) \neq d-1$ and so $d(w', B_{p-1}) = d$.

Choose some vertex in B_{p-1} and call it v'. We know $d(v', w') = d$ so consider some path of length d from v' to w'. Let v^* be the unique point on this path for which $d(v^*, v' = d-2)$. Thus $v^* \notin C_{p-1} \cup D_{p-1}$ and so $v^* \in A_{p-1}$, and also $d(v^*, w') = 2$. Let w'' be some vertex which is adjacent to both v^* and w' so that $d(v', w'') = d-1$. Then because w'' is uncovered (else w' would be dominated), it must be in C_{p-1}. This also means that $v^* \notin B_{p-1}$ by the assumption that $d(B_{p-1}, C_{n-1}) \geq d-1$.

We now move one clump of 2^{d-2} pebbles from v' to v^*, adding one pebble to v^*, which now, by condition 1, has at least two pebbles. We then move two pebbles from v^* and cover w'' with one pebble. We let c_p be the configuration resulting from these moves. We let $D_p = D_{p-1} \cup \{w'\}$ and we again let $B_p = \{v \in G \ : \ c_p(v) \geq 2^{d-2}+1\}$, which just means we have possibly removed v' from B_{p-1}, so $B_p \subseteq B_{p-1}$. If now $c_p(v^*) = 0$, we let $A_p = A_{p-1} \cup \{w''\} \setminus v^*\}$ and $C_p = C_{p-1} \cup \{v^*\} \setminus \{w', w''\}$. Otherwise, if $c_p(v^*) > 0$, let $A_p = A_{p-1} \cup \{w''\}$ and $C_p = C_{p-1} \setminus \{w', w''\}$. This ensures that conditions 1 and 6 still hold for $m = p$. Also, $|C_p| \leq |C_{p-1}| - 1 \leq |C_0| - (p-1) - 1$ and so condition 3 holds for $m = p$. Furthermore, we have used only one clump of 2^{d-2} pebbles, because $v^* \notin B_{p-1}$ and so by using a pebble from v^*, we could not have destroyed a clump. Thus

$$\chi(c_p) = \chi(c_{p-1}) - 2^{d-2} \geq 2^{d-2}(|C_{p-1}| - 1) - 2^{d-2} \geq 2^{d-2}(|C_p| - 1)$$

and therefore condition 2 holds for p. Condition 5 also still holds for $m = p$ because $B_p \subseteq B_{p-1}$ and because we have added only the vertex w' to D_{p-1} and $d(B_{p-1}, w') = d$, so $d(B_{p-1}, D_p) = d$. To see condition 7 is still true, note that to get D_p we have only added w' to D_{p-1}, and certainly, w' is adjacent to w'', which is covered by c_p, so w' is dominated by c_p. Also, the only previously covered vertex of G which is now uncovered is (possibly) v^* but $d(v^*, B_{p-1}) = d-2$, and so v^* is not adjacent to any vertex in D_{p-1} for, by 5, $d(B_{p-1}, D_{p-1}) = d$. Thus, by possibly uncovering v^*, we did not cause any vertex in D_{p-1} to become undominated, so 7 still holds for $m = p$. Finally, the fact that conditions 4 and 8 still hold for $m = p$ is easily seen.

The algorithm continues as long as there is some non-dominated vertex in C_p. By condition 3, it must terminate after at most $|C_0|$ steps, with

$|C_k| = 0$ for some $k \leq |C_0|$. The configuration c_k clearly dominates every vertex of G, and by property 8, c_k is reachable from c by pebbling moves, so c is solvable. $\square$

For $d \geq 3$, Figure 3 shows a graph G which is an example of a graph of diameter d with $n = 2m + d - 1$ vertices for which $\psi(G)$ comes close to the upper bound of $2^{d-2}(n-2) + 1 = 2^{d-1}m + 2^{d-2}(d-3) + 1$.

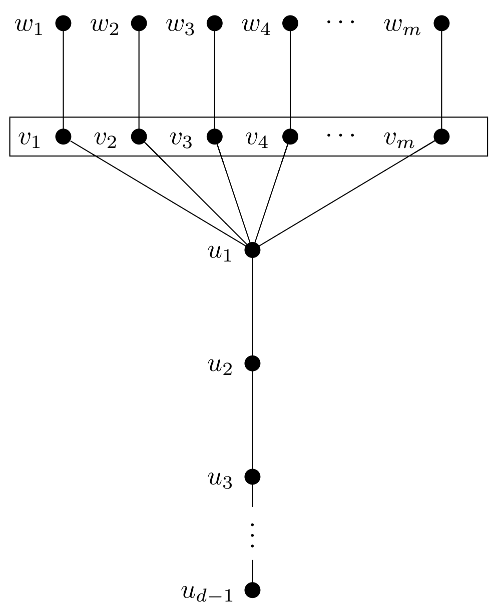

Figure 3: A diameter d graph with high DCP number. The box represents the fact that there is an edge between every pair of vertices inside, making the subgraph induced by $\{v_1, v_2, \ldots, v_m\}$ a complete graph on m vertices.

To dominate vertex w_i, it is easy to see a pebble is needed on w_i or v_i. They each have distance not less than $d-1$ from u_{d-1}, and so it requires 2^{d-1} pebbles on u_{d-1} to supply this pebble. This means at least $2^{d-1}m$ pebbles are needed on u_{d-1} to dominate every w_i, so $\psi(G) \geq 2^{d-1}m$. Further, using the result of [10] and [11], we can calculate $\lambda(G) = 3 \cdot 2^{d-1}m + 2^d - 1$. Clearly, by making m large we can make $\lambda(G)/\psi(G)$ arbitrarily close to 3. Also note that for the complete graph on 2 vertices, $\lambda(G) = 3$ and $\psi(G) = 1$. We conjecture that it is not possible, however, for the ratio to be less than 3:

Conjecture 3.1. *$\lambda(G)/\psi(G) \geq 3$ for all graphs G with more than one vertex.*

4 Subversion DCP

There are several possible generalizations of domination cover pebbling which readily suggest themselves, and many of these are indeed interesting. For instance, we may ask what happens if we simply allow n vertices to remain undominated, that is, if we say a graph has been solved if all but n vertices are dominated by covered vertices. More interestingly, one may relax the requirement that a graph must be dominated by pebbled vertices in order to be solved to the condition that every vertex of a solved graph must have distance no more than n from some pebbled vertex. On the other hand, we could tighten the condition that every vertex of a solved graph is either covered by pebbles or adjacent to a covered vertex by insisting that all vertices, covered or not, must be adjacent to some covered vertex.

However, these generalizations, while natural, may not be different enough from DCP to warrant extensive study. For instance, the problem of diameter bounds seems highly likely to be solvable in each case by an approach quite similar to that in Section 3. Furthermore, in each case, lower bounds which intuitively seem good can be derived from graphs quite similar to the one shown in Figure 3. Therefore, we introduce in this section a less obvious generalization of DCP which we feel makes the analogues to the questions answered in this paper more interesting than they are for the generalizations named above.

Given a graph G and a subset $S \subseteq V(G)$, call the subgraph induced by the set of vertices which are neither in S nor adjacent to a vertex of S the *undominated subgraph* of S. Then we let the *ω-subversion number* of G, denoted $\Omega_\omega(G)$, be the minimum number of pebbles required such that regardless of their initial configuration it is always possible through a sequence of pebbling moves to cover some subset of G that has an undominated subgraph in which there is no component of more than ω vertices.[1] Notice that domination cover pebbling corresponds to the case when $\omega = 0$.

5 Basic Results

We first describe some baseline results for subversion DCP.

Observe that for $\omega \geq 0$, $\Omega_\omega(K_n) = 1$. To see this observe that when any pebble is placed on K_n, the entire graph is dominated.

[1] This definition and the term "subversion" are partly inspired by Cozzens and Wu [4]. Specifically, our parameter ω matches with their use of ω for the order of the largest component of an undominated subgraph.

For $s_1 \geq s_2 \geq \cdots \geq s_r$, let $K_{s_1,s_2,\ldots,s_r}$ be the complete r-partite graph with $s_1, s_2, \ldots, s_r$ vertices in vertex classes $c_1, c_2, \ldots, c_r$ respectively. Then we claim for $\omega \geq 1$, $\Omega_\omega(K_{s_1,s_2,\ldots,s_r}) = 1$. To see this, place a pebble on any vertex in c_i. All the vertices in the other c_i's are dominated. The other vertices in c_1 that are undominated are disjoint from each other. Thus, the claim follows.

Theorem 5.1. *For $\omega \geq 1$, $n \geq \omega + 3$, $\Omega_\omega(W_n) = n - 2 - \omega$, where W_n denotes the wheel graph on n vertices.*

Proof. First, we will show that $\Omega_\omega(W_n) > n - 3 - \omega$. Place a single pebble on each of $n - 3 - \omega$ consecutive outer vertices so that all of the pebbled vertices form a path. This leaves a connected undominated set of size $\omega + 1$. Hence, $\Omega_\omega(W_n) > n - 3 - \omega$. Now, suppose that we place $n - 2 - \omega$ pebbles on W_n. If any vertices have a pair of pebbles on them, the entire graph can be dominated by moving a single pebble to the hub vertex. Hence, each vertex can contain only one pebble. Since every outer vertex is of degree 3, if any vertex is undominated, at least 3 vertices must be dominated but unpebbled. Hence, in order to obtain an undominated set of size $\omega + 1$, there must be $\omega + 4$ vertices that are unpebbled. By the pigeonhole principle, we obtain a contradiction because there are not enough vertices for this constraint to hold. Thus, for $\omega \geq 1$, $n \geq \omega + 3$, $\Omega_\omega(W_n) = n - 2 - \omega$. □

6 Graphs of Diameter 2 and 3

Theorem 6.1. *Let G be a graph of diameter two with n vertices. For $\omega \geq 1$, $\Omega_\omega(G) \leq n - 1 - \omega$.*

Proof. To show that the bound is sharp, consider the graph H_n, defined to be a star graph of order n with ω additional edges added to make the graph induced by one subset of $\omega + 1$ outer vertices connected.

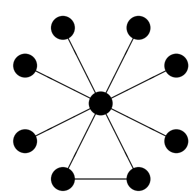

Figure 4: An example of the construction for $n = 9$, $\omega = 1$.

If we place a single pebble on each of the $n - 2 - \omega$ leaves of the star that are not connected to any other outer vertices, the remaining set of undominated vertices is connected and of size $\omega + 1$. Hence, $\Omega(H_n) > n - 2 - \omega$.

Now, let G be a graph of diameter two with n vertices. Suppose there is an arbitrary configuration of pebbles $c(G)$ that contains exactly $n - 1 - \omega$ pebbles. We now show not only that this configuration can be solved to eliminate undominated connected components of order greater than ω, but can in fact be solved such that only at most ω vertices in total are left undominated.

Much as we did in the proof of Theorem 2.1, we let T_1 be the set of vertices $v \in G$ such that $c(v) > 1$, let T_2 be the set vertices $w \in G$ such that $c(w) = 0$ and w is adjacent to some vertex of T_1, and let T_3 be the rest of the vertices, the ones that are neither in T_1 nor adjacent to a vertex of T_1. If $|T_3| \leq \omega$, we are done, because there are no more than ω undominated vertices and thus the largest undominated component has size at most ω. Otherwise, eliminate ω vertices in T_2 from the graph, and consider the induced subgraph G' and the induced configuration c'. We know G' has order $n' = n - \omega$ and c' still has size at least $n - 1 - \omega = n' - 1$. Finally, let $T_1' = T_1$, $T_2' = T_2$ and $T_3' = T_3 \cap V(G')$. The new graph G' may no longer have diameter two, which prevents us from directly applying Theorem 2.1. Nevertheless, we notice that in G', every vertex in T_2' is still adjacent to a vertex in T_1', and every vertex in T_3' is still adjacent to one in T_2'. Also, since in G we know $d(T_1, T_3) = 2$, it follows that no path of length one or two between a vertex in T_1 and another vertex of G can pass through T_3, unless this vertex is the other endpoint. In particular, since the diameter of G is 2, this implies that the shortest path between a vertex in T_1 and another vertex of G cannot pass through a vertex of T_3 as an intermediate vertex, and so the length of the shortest path between a vertex in T_1 and another vertex in G will be unaffected by removing a subset of T_3. This shows that in G', if $s \in T_1'$ and $v \in G'$ then $d(s, v) \leq 2$.

We now note that since we have the right number of pebbles in c' (at least $n' - 1$) we can apply the proof of Theorem 2.1. Following the proof, we see that we will have $S_1 = T_1'$, $S_2 = T_2'$ and $S_3 = T_3'$. Henceforth, the proof never uses the fact that two vertices of the graph have distance at most two from one another except when at least one of the vertices is in S_1. Thus, the algorithm detailed in the proof can be applied *mutatis mutandis* to G', after with G' is dominated by covered vertices. The same sequence of pebbling moves, if performed on G, leaves all vertices except possibly the ω that were eliminated to get G' dominated by covered vertices, thus solving G as desired.

□

In general, however, we believe that determining good diameter bounds for Ω_w will be harder than it is for ψ. It is not even clear to the authors how to construct graphs which establish good lower bounds for large diameters. However, we conclude this section by conjecturing an analogous result for graphs of diameter 3, along with a valid lower-bound construction for this conjecture.

Conjecture 6.1. *Let G be a graph of diameter 3 with n vertices. For $\omega \geq 1$, $\Omega_\omega(G) \leq \lfloor \frac{3}{2}(n-2-\omega)+1 \rfloor$.*

To see that this result, if true, would give a sharp bound, we exhibit a graph G on $n \geq \omega+3$ vertices such that $\Omega_\omega(G) > \lfloor \frac{3}{2}(n-2-\omega) \rfloor$. Take a $K_{\omega+1}$ and attach each of its vertices to some other vertex v. Connect v to each vertex of a $K_{\lceil \frac{n-\omega-2}{2} \rceil}$, call it H. Connect each of the remaining $\lfloor \frac{n-\omega-2}{2} \rfloor$ vertices to a vertex of H, so that each vertex in H has at most one such vertex adjacent to it. Now, place three pebbles on each of the "tendril" vertices attached to H, and if there is one vertex in H without a tendril, place one pebble on it. This is a total of $3\lfloor \frac{n-\omega-2}{2} \rfloor$ (+1 if $n-\omega-2$ is odd) pebbles in this configuration, which is equivalent to $\lfloor \frac{3}{2}(n-2-\omega) \rfloor$. Since it is clearly not possible to dominate the vertices in the $K_{\omega+1}$, the graph still has an undominated component of order $\omega+1$. Thus, $\Omega_\omega(G) > \lfloor \frac{3}{2}(n-2-\omega) \rfloor$.

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