Cover Pebbling Thresholds for the Complete Graph

Anant P. Godbole [1,2]

Department of Mathematics
East Tennessee State University
Johnson City, TN, USA

Nathaniel G. Watson [3]

Department of Mathematics
Washington University in St. Louis
St. Louis, MO, USA

Carl R. Yerger [4]

Department of Mathematics
Harvey Mudd College
Claremont, CA, USA

Abstract

We obtain first-order cover pebbling thresholds of the complete graph for Maxwell Boltzmann and Bose Einstein configurations.

Keywords: cover pebbling, threshold, complete, Maxwell Boltzmann, Bose Einstein

1 Introduction

One recent development in graph theory, suggested by Lagarias and Saks, called *pebbling*, has been the subject of much research and substantive generalizations. It was first introduced into the literature by Chung [2], and has been developed by many others including Hurlbert, who published a survey of pebbling results in [7]. Given a connected graph G, distribute k pebbles on its vertices in some configuration, C. Specifically, a *configuration* of *weight* t on a graph G is a function C from the vertex set $V(G)$ to $\mathbb{N} \cup \{0\}$ such that $\sum_{v \in V(G)} C(v) = t$. A *pebbling move* is defined as the simultaneous removal of two pebbles from some vertex and addition of one pebble on an adjacent vertex. A pebble can be moved to a *root vertex* v if it is possible to place one pebble on v in a sequence of pebbling moves. We define the pebbling number, $\pi(G)$ to be the minimum number of pebbles needed so that for any initial distribution of pebbles, it is possible to move to any root vertex v in G.

The concept of cover solvability, an extension of pebbling, was introduced in [3]. We call a configuration on a graph *cover solvable* if, starting with this configuration, it is possible, through a sequence of pebbling moves, to simultaneously place one pebble on every vertex of the graph. The *cover pebbling number* of a graph, $\gamma(G)$, is defined as the smallest number such that every configuration of this size is cover solvable. One application in [3] for $\gamma(G)$ is based on a military application where troops must be distributed simultaneously.

Another aspect of pebbling that has been explored is the structure one obtains when placing pebbles randomly on graphs. Given a connected graph G, distribute t pebbles on its vertices in some configuration. If the pebbles are indistinguishable, there are $\binom{n+t-1}{t} = \binom{n+t-1}{n-1}$ configurations of t pebbles on n vertices. Using quantum mechanical terminology as in [6], we shall call this situation *Bose Einstein* pebbling and posit that the underlying probability distribution is uniform, i.e. that each of the $\binom{n+t-1}{n-1}$ distributions are equally likely – should the pebbles be thrown randomly onto the vertices. This is the model studied in [4]. Now there is no reason to assume, *a priori*, that the pebbles are indistinguishable. Accordingly, if the pebbles are distinct, we shall refer to our process as *Maxwell Boltzmann* pebbling, in which a random distribution of pebbles leads to n^t equiprobable configurations. Maxwell Boltzmann

[1] Email: godbolea@mail.etsu.edu
[2] This research was supported by NSF Grant DMS-0139286.
[3] Email: meromorphic@gmail.com
[4] Email: cryerger@gmail.com

pebbling seems to have only been studied peripherally in the literature.

Specifically, we seek the probability that a graph G is cover-solvable when t pebbles are placed randomly on it according to the Bose Einstein or Maxwell Boltzmann scheme. Numerous *threshold results* have been determined in [4] for Bose Einstein pebbling (not cover pebbling) of families of graphs such as K_n, the complete graph on n vertices; C_n, the cycle on n vertices; stars; wheels; etc. A threshold result is a theorem of the following kind:

$$t = t_n \gg a_n \Rightarrow \mathbb{P}(G = G_n \text{ is cover solvable}) \to 1 \quad (n \to \infty)$$
$$t = t_n \ll b_n \Rightarrow \mathbb{P}(G = G_n \text{ is cover solvable}) \to 0 \quad (n \to \infty),$$

where we write, for non-negative sequences c_n and d_n, $c_n \gg d_n$ (or $d_n \ll c_n$) if $c_n/d_n \to \infty$ as $n \to \infty$. For the families of complete graphs, wheels and stars, for example, we know ([4]) that $a_n = b_n = \sqrt{n}$ (for pebbling). In many cases, however, the analysis is quite delicate; see [5] for some of the issues involved in finding the pebbling threshold for a family as basic as P_n, the path on n vertices. The fundamental reference [1] contains general results on the existence of sharp pebbling thresholds for families of graphs.

2 Maxwell Boltzmann Cover Pebbling Threshold

Before computing the Maxwell Boltzmann threshold, we will describe a necessary and sufficient condition for finding a cover solution to the complete graph. Suppose that $X = X_{n,t}$ is the number of vertices that contain an odd number of pebbles.

Theorem 2.1 *A configuration of t pebbles on the n vertices of K_n is cover solvable if and only if*

$$X + t \geq 2n$$

With the help of this condition and the second moment method, we can obtain the Maxwell Boltzmann cover pebbling threshold.

Theorem 2.2 *Consider t distinct pebbles that are thrown onto the vertices of the complete graph K_n on n vertices according to the Maxwell Boltzmann distribution. Set $A_0 = 1.5238\ldots$. Then*

$$t = A_0 n + \phi(n)\sqrt{n} \Rightarrow \mathbb{P}(K_n \text{ is cover solvable }) \to 1 \text{ as } n \to \infty.$$

and

$$t = A_0 n - \phi(n)\sqrt{n} \Rightarrow \mathbb{P}(K_n \text{ is cover solvable }) \to 0 \text{ as } n \to \infty.$$

where $\phi(n) \to \infty$ is arbitrary.

3 Bose Einstein Cover Pebbling Threshold

Recall that in Bose Einstein pebbling each of the $\binom{n+t-1}{n-1}$ configurations are just as likely. In this case, there is no obvious sequential process that describes how the pebbles are placed. However, by recasting the problem by Polya sampling and the Azuma-Hoeffding inequality, we can the surprising result that the first order constant for this threshold is the golden ratio.

Theorem 3.1 *Consider t distinct pebbles that are placed on the vertices of the complete graph K_n according to the Bose Einstein distribution. Then, with γ representing the golden ratio $(1+\sqrt{5})/2$,*

$$t = \gamma n + \varphi(n)\sqrt{n} \Rightarrow \mathbb{P}(K_n \text{ is cover solvable}) \to 1 \quad (n \to \infty)$$

and

$$t = \gamma n - \varphi(n)\sqrt{n} \Rightarrow \mathbb{P}(K_n \text{ is cover solvable}) \to 0 \quad (n \to \infty),$$

where $\varphi(n) \to \infty$ is arbitrary.

References

[1] Bekmetjev, A., G. Brightwell, A. Czygrinow, and G. Hurlbert, *Thresholds for families of multisets, with an application to graph pebbling* Discrete Math. **269** (2003), 21–34.

[2] Chung, F., *Pebbling in hypercubes* SIAM J. Discrete Mathematics. **2** (1989), 467–472.

[3] Crull, B., T. Cundiff, P. Feltman, G. H. Hurlbert, L. Pudwell, Z. Szaniszlo, Z. Tuza, *The Cover Pebbling Number of Graphs*, (2004), submitted.

[4] Czygrinow, A., N. Eaton, G. Hurlbert, and P. M. Kayll, *On pebbling thresholds for graph sequences*, Discrete Math. **247** (2002), 93–105.

[5] Czygrinow, A., and G. Hurlbert, *Toward the pebbling threshold of paths*, (2004), preprint.

[6] Feller, W., *An Introduction to Probability Theory and its Applications, Volume 1, 3rd Edition*, John Wiley, New York, 1968.

[7] Hurlbert, G., *A survey of graph pebbling*, Congr. Numer. **139** (1999), 41–64.

[8] Steele, J. M., *Probability Theory and Combinatorial Optimization*, NSF-CBMS Regional Research Conference Lecture Notes Series, SIAM, Philadelphia, **69**, 1997.