Combinatorial Interpretations of Spanning Tree Identities

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Abstract

We present a combinatorial proof that the wheel graph W_n has $L_{2n} - 2$ spanning trees, where L_n is the nth Lucas number, and that the number of spanning trees of a related graph is a Fibonacci number. Our proofs avoid the use of induction, determinants, or the matrix tree theorem.

1 Introduction

Let G be a graph and let $\tau(G)$ be the number of spanning trees of G. In this paper we will present combinatorial proofs that determine $\tau(G)$ for the wheel graph and a related auxiliary graph. Two simple bijections will provide a direct explanation as to why the number of spanning trees for these graphs are Fibonacci and Lucas numbers.

Definition 1.1. *For $n \geq 1$, The wheel graph W_n has $n+1$ vertices, consisting of a cycle of n outer vertices, labelled $w_1, \ldots, w_n$, and a "hub" center vertex, labelled w_0, that is adjacent to all the n outer vertices .*

For example, W_8 is presented in Figure 1. The *Lucas numbers* are recursively defined by $L_1 = 1$, $L_2 = 3$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 3$.

Theorem 1.2. *For $n \geq 1$, $\tau(W_n) = L_{2n} - 2$.*

This result was first proved by Sedlacek in [5] and later by Myers in [3]. As part of Myers' proof, he employs an auxiliary graph, denoted by A_n, that is similar to the wheel graph and presented in Figure 2. For $n \geq 2$, A_n has $n+1$ vertices and $2n+1$ edges, consisting of a path of n outer vertices, labelled

Figure 1: The wheel graph W_8.

$a_1, \ldots, a_n$, and a hub vertex a_0 that is adjacent to all n outer vertices. In addition, a_0 has an extra edge connecting to a_1 and an extra edge connecting to a_n. We label the two edges from a_0 to a_1 as red and blue, and do the same for the edges from a_0 to a_n. Let f_n denote the nth Fibonacci number with initial conditions $f_1 = 1$, and $f_2 = 2$.

Theorem 1.3. *For $n \geq 2$, $\tau(A_n) = f_{2n+1}$.*

Figure 2: The auxiliary graph A_8.

One way to determine $\tau(A_n)$, as shown by Koshy [2], is to apply the matrix tree theorem [6], first proved by Kirchhoff, by computing the determinant of the n-by-n tridiagonal matrix

$$A_n = \begin{vmatrix} 3 & -1 & 0 & \ldots & & 0 \\ -1 & 3 & -1 & \ldots & & 0 \\ 0 & -1 & 3 & \ldots & & 0 \\ & & & \vdots & & -1 \\ 0 & 0 & 0 & \ldots & -1 & 3 \end{vmatrix}.$$

Expanding along the first row, and proceeding inductively, it follows that $\tau(A_n) = |A_n| = 3|A_{n-1}| - |A_{n-2}| = 3f_{2n-1} - f_{2n-3} = f_{2n+1}$.

The matrix tree theorem also indicates that $\tau(W_n)$ equals the determinant of the following matrix n-by-n circulant matrix

$$B_n = \begin{vmatrix} 3 & -1 & 0 & \dots & & -1 \\ -1 & 3 & -1 & \dots & & 0 \\ 0 & -1 & 3 & \dots & & 0 \\ & & & \vdots & & -1 \\ -1 & 0 & 0 & \dots & -1 & 3 \end{vmatrix}.$$

Expanding $|B_n|$ along its first row, we obtain $|A_n|$ as one of its subdeterminants. Proceeding by induction and with a bit more computation (see [2]), $\tau(W_n) = L_{2n} - 2$ can then be obtained. In the next two sections, we give combinatorial proofs of Theorems 1.2 and 1.3 that are much more direct.

2 Combinatorial Proof of $\tau(W_n) = L_{2n} - 2$

The Lucas number L_n counts the ways to tile a bracelet of length n and width 1 using 1×1 squares and 1×2 dominoes [1]. Equivalently, L_n is the number of matchings in the cycle graph C_n. Observe that *even* cycle graphs C_{2n} have exactly two perfect matchings and thus $L_{2n} - 2$ *imperfect* matchings, such as the one in Figure 3.

Figure 3: An imperfect matching of C_8.

Given an imperfect matching M (a subgraph of C_{2n} where every vertex c_i has degree 0 or 1), we construct a spanning tree T_M of W_n as follows:

1. For $1 \le i \le n$, an edge exists from w_0 to w_i if and only if c_{2i-1} has degree 0 in M.

2. For $1 \le i \le n$, an edge exists from w_i to w_{i+1} (where w_{n+1} is identified with w_1) if and only if c_{2i} has degree 1 in M.

The bijection is illustrated in Figure 4.

Figure 4: An example of the bijection for $n = 4$.

To see that T_M is a spanning tree of W_n, suppose that M has x vertices of degree 1 and y vertices of degree 0; thus $x + y = 2n$. Observe that vertices of degree 1 come in adjacent pairs and that if v_j has degree 0, then the next vertex of degree 0, clockwise from v_j, must be v_k, where k and j have opposite parity. Thus, T_M will use exactly $x/2 + y/2 = n$ edges of W_n. Since W_n has $n + 1$ vertices, we need only show that T_M has no cycles. Suppose, to the contrary, that T_M has a cycle C. Then C, denoted by $w_0 w_i w_{i+1} \cdots w_k w_0$, must use two edges adjacent to w_0 (otherwise M would be a *perfect* matching). Thus, c_{2i-1} and c_{2k-1} have degree 0 in M and hence some vertex c_{2j} must also have degree 0 where c_{2j} is strictly between c_{2i-1} and c_{2k-1} on C. But since c_{2j} has degree 0, there is no edge in T_M from w_j to w_{j+1}, a contradiction. Hence no cycle C exists on T_M and so T_M is a tree.

The process is reversible since a spanning tree T of W_n completely determines the degree $d_k \in \{0, 1\}$ of each vertex c_k in a subgraph of C_{2n}. Since w_0 is not an isolated vertex of T, not all d_k are equal to 1. We show that C_{2n} has a unique matching that satisfies this degree sequence by showing that every string of 1s has even length; i.e., if $d_k = 0$, $d_{k+1} = d_{k+2} = \cdots = d_{k+j} = 1$, and $d_{k+j+1} = 0$, then j must be even. For if $k = 2i - 1$ is odd and j is odd then the tree T would contain a cycle $w_0 w_i w_{i+1} \cdots w_{i+(j+1)/2} w_0$. If $k = 2i$ is even and j is odd, then T is not connected since the path $w_{i+1} w_{i+2} \cdots w_{i+(j+1)/2}$ is disconnected from the rest of T.

3 Combinatorial Proof of $\tau(A_n) = f_{2n+1}$

The Fibonacci number f_n counts the ways to tile a $1 \times n$ rectangle using 1×1 squares and 1×2 dominoes [1]. Alternatively, f_n counts the matchings of P_n, the path graph on n vertices, whose vertices are consecutively denoted $p_1, \ldots, p_n$. Let M be an arbitrary matching of P_{2n+1}. We construct a spanning tree T_M of A_n as follows:

1. For $1 \leq i \leq n$, T_M has an edge from from a_0 to a_i if and only if vertex p_{2i} has degree 0 in M. (For $i = 1$ or n, then this refers to the red edge.)
2. For $0 \leq i \leq n-1$, T_m has an edge from a_i to a_{i+1} if and only if p_{2i+1} has degree 1 in M. (For $i = 0$, this refers to the blue edge.)
3. T_M has a blue edge from a_0 to a_n if and only if p_{2n+1} has degree 1 in M.

Notice that these rules make it impossible for T_M to contain two edges from a_0 to a_1 or two edges from a_0 to a_n. The bijection is illustrated in Figure 5

Figure 5: An example of the bijection for $n = 4$.

Like before, we prove that T_M is a spanning tree of A_n. Suppose that M has a and b vertices of degree 0 and 1 respectively; thus $a + b = 2n + 1$. Reasoning as before, M has $b/2$ odd vertices of degree 1 and $(a-1)/2$ even vertices of degree 0. Thus, T_M has $(a-1)/2 + b/2 = n$ edges. Suppose for the sake of contradiction, that T_M has a cycle C. Then C, denoted by $a_0 a_i a_{i+1} \cdots a_k a_0$, must use two edges adjacent to a_0. Thus p_{2i} and p_{2k} have degree 0 in M and hence some vertex p_{2j+1} must also have degree 0 where p_{2j+1} is strictly between p_{2i} and p_{2k} on C. But since p_{2j+1} has degree 0, there is no edge in T_M from a_j to a_{j+1}, a contradiction. Hence no cycle C exists on T_M and so T_M is a tree.

The process is also reversible since a spanning tree T of A_n completely determines the degree $d_k \in \{0, 1\}$ of each vertex p_k in a subgraph of P_{2n+1}. Again, not all d_k are equal to 1, since T would contain the cycle $a_0 a_1 \cdots a_n a_0$.

To prove that P_{2n+1} has a unique matching that satisfies this degree sequence, suppose that for some k, j, $d_k = 0$, $d_{k+1} = d_{k+2} = \cdots = d_{k+j} = 1$, and $d_{k+j+1} = 0$. As before, if $k = 2i$ is even and j is odd, then the tree T contains the cycle $a_0 a_i a_{i+1} \cdots a_{i+(j+1)/2} a_0$. If $k = 2i - 1$ is odd and j is odd, then T is not connected since the path $a_i a_{i+1} \cdots a_{i+(j-1)/2}$ is disconnected from the rest of T. Thus j must be even, and the matching generating T is unique.

References

[1] A. T. Benjamin and J. J. Quinn. *Proofs that Really Count: The Art of Combinatorial Proof*, Mathematical Association of America (2003).

[2] T. Koshy, *Fibonacci and Lucas Numbers With Applications*, Wiley and Sons (2001).

[3] B. R. Myers, Number of Spanning Trees in a Wheel, *IEEE Transactions on Circuit Theory*, CT-18 (1971) 280-282.

[4] K. R. Rebman, The Sequence: 1 5 16 45 121 320 ... in Combinatorics, *The Fibonacci Quarterly*, 13:1 (1975) 51-55.

[5] J. Sedlacek, On the Skeletons of a Graph or Digraph, *Proceedings of the Calgary International Conference of Combinatorial Structures and Their Applications*, Gordon and Breach (1970) 387-391.

[6] D. West, *Introduction to Graph Theory, Second Edition*, Prentice Hall (2002).

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