CATALAN DETERMINANTS — A COMBINATORIAL APPROACH

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Abstract. Determinants of matrices involving the Catalan sequence have appeared throughout the literature. In this paper, we focus on the evaluation of Hankel determinants featuring Catalan numbers by counting nonintersecting path systems in an associated Catalan digraph. We apply this approach in order to revisit and extend a result due to Cvetkovic-Rajkovic-Ivkovic, where we find that Hankel determinants involving the sum of successive Catalan numbers produce Fibonacci sequences.

1. Introduction

Combinatorial interpretations of determinants can bring deeper understanding to their evaluations; this is especially true when the entries of a matrix have natural graph theoretic descriptions. Lindström, Gessel, and Viennot [5, 6] reveal how the determinant counts signed nonintersecting path-systems in an associated directed graph. For an $n \times n$ matrix $A = (a_{ij})$, the general idea is to create an acyclic directed graph D with n origin nodes, $o_1, o_2, \ldots, o_n$, and n destination nodes, $d_1, d_2, \ldots, d_n$, so that the number of paths from origin o_i to destination d_j is a_{ij}. Given a permutation σ in S_n, the product $\prod_{i=1}^{n} a_{i\sigma(i)}$ counts the ways to construct n directed paths in D where the i^{th} path goes from origin o_i to destination $d_{\sigma(i)}$. We call such a system of n directed paths an *n-route*. Since $\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$, where $\operatorname{sgn}(\sigma)$ is the sign of the permutation, the determinant is the number of n-routes induced by even permutations (called *even n-routes*) minus the number of n-routes induced by odd permutations (called *odd n-routes*). A sign reversing involution exists between even and odd n-routes provided some vertex of D is shared by two paths in the route, i.e. whenever two paths intersect. So calculating the determinant reduces to determining the number of even *nonintersecting* n-routes minus the number of odd *nonintersecting* n-routes.

When matrix entries are binomial coefficients, Fibonacci numbers, or a combination thereof, the nonintersecting path interpretation leads to insightful evaluations [1, 2, 10]. Catalan numbers have natural interpretations as lattice paths; consequently matrices with Catalan entries also have beautiful combinatorial explanations. Recall that the n^{th} Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ counts the number of paths connecting $(0,0)$ to (n,n) that travel along the grid of integer lattice points of $\mathbf{R}^2$ where each path moves up or right in one-unit steps and no path extends above the line $y = x$ [9]. This interpretation is key to applying nonintersecting n-route

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arguments to matrices containing Catalan numbers. In Section 2 we explore Mays and Wojciechowski's work [7] calculating the determinant of the Hankel matrix $M_n^t = (C_{i+j+t})_{i,j=0}^{n-1}$ to better familiarize the reader with the proof technique. In Section 3 we extend the ideas to calculate determinants when the matrix entries are sums of successive Catalan numbers and the determinants contain Fibonacci numbers.

2. Hankel Matrices of Catalan Numbers

Given $n \geq 1$ and $t \geq 0$, define D_n^t, the *Catalan digraph with n origins and destinations at distance t*, to contain the vertices of the integer lattice on and below the line $y = x$ with arcs oriented to the right and up and origins $o_i = (-i, -i)$ and destinations $d_i = (i+t, i+t)$ for $i = 0, 1, \ldots, n-1$. See Figure 1. Notice for $0 \leq i, j \leq n-1$, the number of directed paths from origin o_i to destination d_j is C_{t+i+j}.

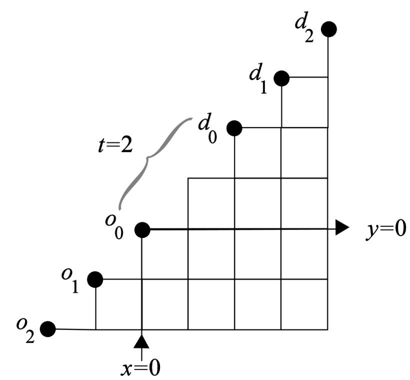

Figure 1. The Catalan digraph, D_3^2. Arcs are oriented to the right and up.

Mays and Wojciechowski [7] directly argue that the determinant of the matrix

$$M_n^t = \begin{bmatrix} C_t & C_{t+1} & \cdots & C_{t+n-1} \\ C_{t+1} & C_{t+2} & \cdots & C_{t+n} \\ \vdots & & \ddots & \vdots \\ C_{t+n-1} & C_{t+n} & \cdots & C_{t+2n-2} \end{bmatrix}$$

equals the number of nonintersecting n-routes corresponding to the identity permutation on the Catalan digraph D_n^t. Because of its structure, the only nonintersecting n-routes in D_n^t correspond to the identity permutation; this leads to a clear understanding of the following determinants, which appear in [7] and we present here to motivate the new results in Section 3.

Identity 1. *For $n \geq 1$, $\det(M_n^0) = 1$ and $\det(M_n^1) = 1$.*

When o_0 and d_0 coincide ($t = 0$) or their x- and y-coordinates differ by one ($t = 1$), the only nonintersecting n-route in the digraph is a collection of nested right angles.

Identity 2. *For $n \geq 1$, $\det(M_n^2) = n + 1$.*

Any nonintersecting n-route in D_n^2 uses exactly n of the $n+1$ possible lattice points along the line $y = 2 - x$. Selecting which of these $n+1$ points to avoid uniquely determines a nonintersecting n-route.

Identity 3. *For $n \geq 1$, $\det(M_n^3) = \sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+2)(2n+3)}{6}$.*

Any nonintersecting n-route in D_n^3 uses exactly n of the $n+1$ possible lattice points $\{(k+1, -k+2) : 1 \leq k \leq n+1\}$ along the line $y = 3 - x$. For a given value of k, avoiding the point $(k+1, -k+2)$ in an nonintersecting n-route requires that the directed paths from origins $o_{k-1}, o_k, \ldots, o_{n-1}$ use horizontal arcs until the line $y = 3 - x$ and vertical arcs thereafter. The remaining $k-1$ paths from origins $o_0, o_1, \ldots, o_{k-2}$ use exactly $k-1$ of the k possible lattice points along the line $y = 2 - x$. The same is true for the line $y = 4 - x$. Selecting the points to avoid on each line, completely determines a nonintersecting n-route. See Figure 2. Since k ranges between 1 and $n+1$, the determinant equals $\sum_{k=1}^{n+1} k^2$.

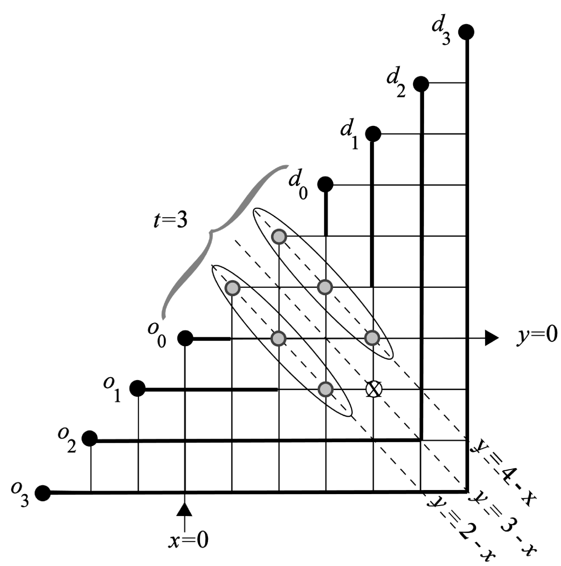

FIGURE 2. There are 3^2 nonintersecting 4-routes in D_4^3 avoiding the point $(4, -1)$. A nonintersecting 4-route is determined by selecting one of three points on each line $y = 2 - x$ and $y = 4 - x$ to avoid.

3. Hankel Matrices of Catalan Sums

The next step is to consider the determinants of Hankel matrices containing the sums of consecutive Catalan numbers. Let $S_n^t = (C_{i+j+t} + C_{i+j+t+1})_{i,j=0}^{n-1}$. So

$$S_n^t = \begin{bmatrix} C_t + C_{t+1} & C_{t+1} + C_{t+2} & \cdots & C_{t+n-1} + C_{t+n} \\ C_{t+1} + C_{t+2} & C_{t+2} + C_{t+3} & \cdots & C_{t+n} + C_{t+n+1} \\ \vdots & \vdots & \ddots & \vdots \\ C_{t+n-1} + C_{t+n} & C_{t+n} + C_{t+n+1} & \cdots & C_{t+2n-2} + C_{t+2n-1} \end{bmatrix}.$$

Using Hankel transforms and generating functions, Cvetković, Rajković, and Ivković [4] showed that $\det(S_n^0) = f_{2n}$ and $\det(S_n^1) = f_{2n+1}$ where $f_0 = 1$, $f_1 = 1$, and for $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$. The simplicity of these answers begs for an elegant combinatorial solution. To see why $\det(S_n^t)$ is a Fibonacci number for $t = 0$ or 1, we present a bijection between nonintersecting n-routes in an associated digraph and tilings of a rectangle with squares and dominoes. Recall that for $n \geq 0$, the Fibonacci number f_n counts the ways to tile a $1 \times n$ rectangle using 1×1 squares and 1×2 dominoes [3, 8]. In fact, using this bijection with the ideas in Section 2 will allow us to take the results further and calculate $\det(S_n^t)$ when $t = 2$.

We begin by defining a digraph $\tilde{D}_n^t$ whose signed sum of nonintersecting n-routes calculates $\det(S_n^t)$ for $t \geq 1$. This directed graph is obtained by adding n additional vertices, $2n$ arcs, and relocating the destinations in the Catalan digraph D_n^t. Specifically, $\tilde{D}_n^t$ is the digraph consisting of all the vertices of D_n^t plus the set of vertices $\{(i+t, i+t+1)\} : 0 \leq i \leq n-1\}$ and the arcs of D_n^t plus vertical arcs $\{((i+t, i+t), (i+t, i+t+1)) :$ for $0 \leq i \leq n-1\}$ and horizontal arcs $\{((i+t+1, i+t+1), (i+t, i+t+1)) :$ for $0 \leq i \leq n-1\}$. Notice, that the additional horizontal arcs represent steps to the left as opposed to the usual steps to the right in the Catalan digraph. Finally for $0 \leq i \leq n-1$, we preserve origin $o_i = (-i, -i)$ and relocate destination d_i to the newly added vertex $(i+t, i+t+1)$. See Figure 3. Then the number of directed paths from origin o_i to destination d_j is $C_{t+i+j} + C_{t+i+j+1}$ (C_{t+i+j} ways when the final step is upward and $C_{t+i+j+1}$ ways when the final step is to the left). It is easy to see that a nonintersecting n-route in $\tilde{D}_n^t$ can only arise from the identity permutation.

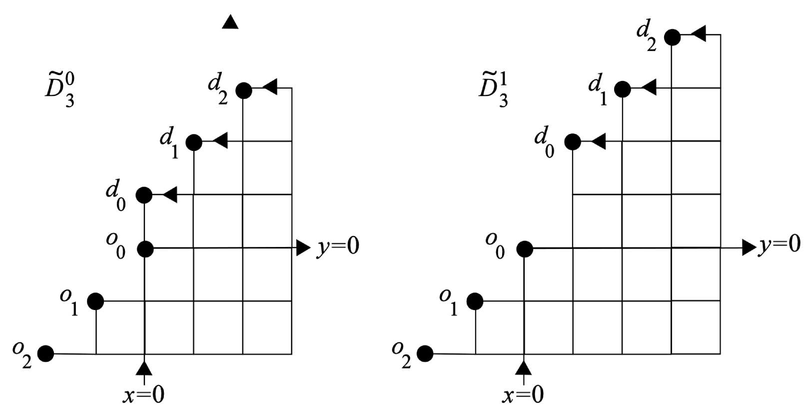

FIGURE 3. The Modified Catalan digraphs, $\tilde{D}_3^0$ and $\tilde{D}_3^1$. Unmarked arcs are oriented to the right and up.

In the first identity, we consider the case when $t = 0$ where destination d_0 lies one step above origin o_0.

Identity 4. *For* $n \geq 1$, $\det(S_n^0) = f_{2n}$.

To understand this identity, we create a bijection between nonintersecting n-routes of $\tilde{D}_n^0$ and Fibonacci tilings of a $1 \times 2n$ board with squares and dominoes. More precisely, each nonintersecting n-route with exactly k paths taking final steps to the left is mapped to a Fibonacci tiling of a $1 \times 2n$ board containing exactly k dominoes.

Notice that for $i = 0, \ldots, n-1$ a path from o_i to d_i in a nonintersecting n-route will either be "L-shaped," taking $2i$ steps to the right, followed by $2i+1$ vertical steps (and we label this path with $\ell(i) = 0$) or it will take $2i$ steps to the right, followed by k vertical steps for some $0 \leq k \leq 2i$. Then it completes the "hook" by taking one step right, $2i+1-k$ steps up, and one final step to the left. We label this path with $\ell(i) = 2i+1-k$. Hence, when $\ell(i) \neq 0$, $\ell(i)$ is the vertical distance between the horizontal step from $x = i$ to $x = i+1$ and the final left step $((i+1, i+1), (i, i+1))$. In Figure 4, $\ell(0) = 0$, $\ell(1) = 2$, and $\ell(2) = 4$. When a path in a nonintersecting n-route of $\tilde{D}_n^0$ begins at o_i and ends with a left-step (into d_i), the paths from origins $o_{i+1}, o_{i+2}, \ldots, o_{n-1}$ must end with left-steps as well since the vertical passage to their intended destinations are sequentially blocked. Notice that two consecutive nonzero values $\ell(i)$ and $\ell(i+1)$ must differ by at least two.

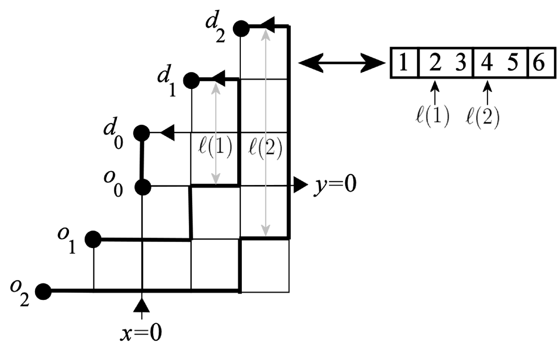

FIGURE 4. The nonintersecting 3-route above with $\ell(1) = 2, \ell(2) = 4$ is mapped to the tiling of length 6 with dominoes beginning on cells 2 and 4.

A nonintersecting n-route will have, for some $0 \leq k \leq n$, $n-k$ L-shaped paths from o_i to d_i (for $i = 0, \ldots, n-k-1$) followed by k hooked paths from o_i to d_i (for $i = n-k, \ldots, n-1$). We map such an n-route to the Fibonacci tiling with k dominoes beginning on cells $\ell(n-k), \ell(n-k+1), \ldots, \ell(n-1)$ and squares everywhere else. Since $0 \leq \ell(n-1) \leq 2n-1$, a final domino can be begin on cell $2n-1$ and the Fibonacci tiling has length $2n$.

Every nonintersecting n-route is mapped to a unique tiling of the $1 \times 2n$ rectangle. Conversely, every tiling of a $1 \times 2n$ rectangle (with k dominoes) induces a unique nonintersecting n-route since the position of the dominoes defines the locations of the final steps to the right for (the last k) paths ending with left-steps. Thus, a bijection exists and $\det(S_n^0)$ equals the number of Fibonacci tilings of length $2n$, namely f_{2n}.

Identity 5. *For* $n \geq 1$, $\det(S_n^1) = f_{2n+1}$.

The bijection is essentially the same as above. A nonintersecting n-route in $\tilde{D}_n^1$ whose k paths from origins $o_{n-k}, o_{n-k+1}, \ldots, o_{n-1}$ utilize the final left-steps is mapped to the tiling with dominoes beginning on cells $\ell(n-k), \ell(n-k+1), \ldots, \ell(n-1)$ and squares everywhere else. Here, $0 \leq \ell(n-1) \leq 2n$, so a final domino can begin on cell $2n$ and the Fibonacci tiling has length $2n+1$.

Identity 6. *For $n \geq 1$, $\det(S_n^2) = (n+1)f_{2n+2} - f_{2n+1}$.*

Similar to the argument for Identity 2, any nonintersecting n-route in $\tilde{D}_n^2$ uses exactly n of the $n+1$ possible lattice points $\{(k, 2-k) : 1 \leq k \leq n+1\}$ along the line $y = 2-x$. For a given value of k, a nonintersecting n-route avoiding the point $(k, 2-k)$ is uniquely determined between the origins and the line $y = 2-x$. The completion of the n-route describes a Fibonacci tiling of length $2n+2$ as previously described in Identity 4. The Fibonacci tilings that occur depend on the value of k; specifically, all $(2n+2)$-tilings occur except for those ending with $n-k+2$ dominoes. To see why, consider the consequence of a $(2n+2)$-tiling ending with $n-k+2$ dominoes. The corresponding n-route would have $\ell(n-1) = 2n+1$, $\ell(n-2) = 2n-1, \ldots, \ell(k-2) = 2k-1$, forcing the n-route to pass through the forbidden point $(k, 2-k)$. See Figure 5. When $k = 1$, we avoid the tiling containing $n+1$ dominoes since each domino corresponds to a left-stepping path but the route only contains n paths. When $k = n+1$, we avoid all tilings ending in a domino. So given a value of k, there are f_{2k-2} excluded tilings. Thus the total number of nonintersecting paths is $\sum_{k=1}^{n+1}(f_{2n+2} - f_{2k-2}) = (n+1)f_{2n+2} - f_{2n+1}$ since, by telescoping sums or combinatorial argument [3], the sum of the first $n+1$ even-indexed Fibonacci numbers equals f_{2n+1}.

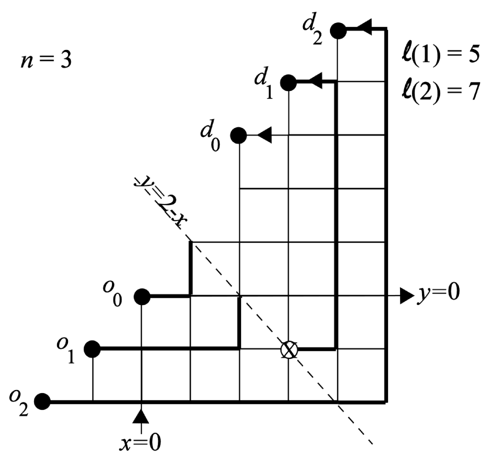

Figure 5. For a nonintersecting 3-route of $\tilde{D}_3^2$ to bypass the lattice point $(3, -1)$, it must not have $\ell(1) = 5$ and $\ell(2) = 7$ as shown above. Consequently all Fibonacci tilings of a 1×8 board ending with two dominoes are removed from consideration.

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