Geometric and Harmonic Variations of the Fibonacci Sequence

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First told by Fibonacci himself, the story that often accompanies one's initial encounter with the sequence *1,1,2,3,5,8,...* describes the size of a population of rabbits. The original question concerns the number of pairs of rabbits there are in a population; for simplicity we consider individual rabbits rather than pairs. In general, a rabbit is born in one season, grows up in the next, and in each successive season gives birth to one baby rabbit. Here, the sequence $\{f_n\}$ that enumerates the number of births in each season is given by $f_{n+2} = f_{n+1} + f_n$ for $n \geq 1$, with $f_1 = f_2 = 1$, which coincides precisely with the Fibonacci sequence. Also, recall that the asymptotic exponential growth rate of the Fibonacci numbers equals the golden ratio, $\frac{1+\sqrt{5}}{2}$. Further discussion of this golden ratio can be found in [1]. In addition, there is a very large amount of literature on the Fibonacci sequence, including the *Fibonacci Quarterly*, a journal entirely devoted to the Fibonacci sequence and its extensions.

In this article, we consider similar recurrences and examine their asymptotic properties. One way this has been previously studied is by defining a new sequence, $G_{n+r} = \alpha_1 G_{n+r-1} + \alpha_2 G_{n+r-2} + \cdots + \alpha_r G_n$ for $n \geq 1$, and giving a set of initial conditions $\{G_1, G_2, ..., G_r\}$. Other modifications include a non-deterministic version that allows for randomness in the values of the terms of the sequence, while still having successive terms depend on the previous two: one such recurrence is given by $t_{n+2} = \alpha_{n+2} t_{n+1} + \beta_{n+2} t_n$ where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of random variables distributed over some subset of the real numbers. In the case when $\{\alpha_n\}$ and $\{\beta_n\}$ are independent Rademachers (symmetric Bernoullis), that is, each taking values ± 1 with equal probability, Divakar Viswanath showed that although the terms of $\{t_n\}$ are random, asymptotically the sequence experiences exponential growth almost surely; $\sqrt[n]{|t_n|}$ approaches a constant 1.1319... as $n \to \infty$ [6]. Building from this result, Mark Embree and Lloyd Trefethen determined the asymptotic growth rate when α_n and β_n take the form of other random variables [2]. In this article, we determine the growth rates of other variations of the Fibonacci sequence, specifically those we call the geometric and harmonic Fibonacci sequences.

The Geometric and Harmonic Fibonacci Sequences

There has been significant study of Fibonacci-like sequences that are linear, that is, recurrence relations of the form given by $\{G_n\}$ defined above. In this paper, though, we will consider two *non-linear* Fibonacci recurrences. First, note that we can view the Fibonacci sequence as a recurrence in which each term is twice the *arithmetic* mean of the two previous terms. In this light, we introduce the *geometric* Fibonacci sequence $\{g_n\}$ and the *harmonic* Fibonacci sequence $\{h_n\}$, in which each successive term is twice the geometric or harmonic mean, respectively, of the previous two terms in the sequence. That is, we define

$$g_{n+2} = 2\sqrt{g_{n+1}g_n} \text{ for } n \geq 1, \text{ with } g_1 = g_2 = 1,$$

and

$$h_{n+2} = \frac{4}{\frac{1}{h_{n+1}} + \frac{1}{h_n}} \text{ for } n \geq 1, \text{ with } h_1 = h_2 = 1.$$

We motivate the study of the geometric and harmonic sequences by a desire to examine properties associated with the triumvirate of the arithmetic, geometric, and harmonic means.

Term #	Fibonacci	Geometric Fibonacci	Harmonic Fibonacci
1	1	$1 = 2^0$	1
2	1	$1 = 2^0$	1
3	2	$2 = 2^1$	2
4	3	$2.828\ldots = 2^{3/2}$	$2.666\ldots = \frac{8}{3}$
5	5	$4.756\ldots = 2^{9/4}$	$4.571\ldots = \frac{32}{7}$
6	8	$7.336\ldots = 2^{23/8}$	$6.736\ldots = \frac{128}{19}$
7	13	$11.814\ldots = 2^{57/16}$	$10.893\ldots = \frac{512}{47}$
8	21	$18.619\ldots = 2^{135/32}$	$16.650\ldots = \frac{2048}{123}$

Table 1: Here are the first eight terms of each Fibonacci sequence.

Arithmetic-Geometric-Harmonic Mean Relations

The first historical reference to the arithmetic, geometric and harmonic means is attributed to the school of Pythagoras, where it was applied to both mathematics and music. Initially dubbed the subcontrary mean, the harmonic mean acquired its current name because it relates to "the 'geometrical harmony' of the cube, which has 12 edges, 8 vertices, and 6 faces, and 8 is the mean between 12 and 6 in the theory of harmonics" [4]. Today, the harmonic mean has direct applications in such fields as physics, where it is used in circuits and in optics (through the well-known lens-makers' formula).

We also know that the following hierarchy always holds: the arithmetic mean of two non-negative numbers is always at least as great as their geometric mean, which in turn is at least as great as the harmonic mean. That is, given two numbers a and b, $\frac{a+b}{2} \geq \sqrt{ab} \geq \frac{2}{\frac{1}{a}+\frac{1}{b}}$.

As a result of the arithmetic-geometric-harmonic mean inequalities, the terms of the corresponding sequences we defined satisfy the inequality $f_n \geq g_n \geq h_n$ for all n. Next, we will see that the asymptotic *growth rates* of the Fibonacci sequence, along with those of our geometric and harmonic variations of the sequence, exist and also satisfy this inequality.

Calculating the Growth Rates for the Geometric and Harmonic Fibonacci Sequences

In order to solve the difference equations for $\{g_n\}$ and $\{h_n\}$, we will proceed in the same manner as solving a non-homogeneous differential equation. First, we will define a characteristic equation for the recurrence from which we can obtain a homogeneous solution. Then, using the roots of the characteristic equation, we will apply the method of undetermined coefficients to obtain a particular solution (if necessary), which when combined with the homogeneous solution and the initial conditions yields a solution to the difference equation.

As a first example, we will derive the growth rate for the Fibonacci sequence in this manner. Our characteristic equation of the recursive sequence $\{f_n\}$ defined by $f_{n+2} = f_{n+1} + f_n$, is $x^2 - x - 1 = 0$. This has solutions of $x = \frac{1 \pm \sqrt{5}}{2}$. So, our homogeneous solution is $f_n = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$. Using our two initial conditions of the Fibonacci sequence, namely $f_1 = 1, f_2 = 1$, we see that $c_1 = \frac{1}{\sqrt{5}}$ and $c_2 = \frac{-1}{\sqrt{5}}$. This gives a general form (Binet's formula) for the nth Fibonacci number as $f_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n \sqrt{5}}$. Thus, we have an asymptotic bound of $\frac{1+\sqrt{5}}{2}$, as desired.

Next, we consider our *geometric* Fibonacci sequence $\{g_n\}$ as defined above and proceed to determine its growth rate (note, though that by inspection it is not entirely clear that an asymptotic growth rate exists). A naive way to guess what this rate is results from the following steps. If we assume that this asymptotic growth rate exists, we can determine the limit of the ratio of successive terms in the geometric mean recurrence directly from the recurrence relations. Let R_g be the asymptotic growth rate, $R_g = \lim_{n \to \infty} \frac{g_{n+1}}{g_n}$. Next, we solve for R_g:

$$g_{n+2} = 2\sqrt{g_{n+1} g_n}$$

$$\Rightarrow (g_{n+2})^2 = 4g_{n+1}g_n$$

$$\Rightarrow \lim_{n\to\infty} \frac{{g_{n+2}}^2}{{g_{n+1}}^2} = 4\lim_{n\to\infty} \frac{g_n}{g_{n+1}}$$

$$\Rightarrow {R_g}^2 = 4\frac{1}{R_g}$$

$$\Rightarrow R_g = 4^{1/3}$$

From this calculation emerges the surprising result that the asymptotic growth rate of our geometric Fibonacci sequence is likely to be the cube root of four.

To obtain this result in a more rigorous manner, we instead solve for a closed-form expression; from this expression, the growth rate is shown to exist and indeed equal $4^{1/3}$. The most common method for solving this form of recursive relation is by using generating functions; for example, the asymptotic growth rate of the regular Fibonacci sequence, which interestingly is the golden ratio $\frac{1+\sqrt{5}}{2} = 1.6180...$ (c.f., [3]), can be found in this way. Here we use a different technique–the one described above–that, in this case, simplifies calculations. Recall that we have the following relation for our geometric Fibonacci sequence: $g_{n+2} = 2\sqrt{g_{n+1}g_n}$. Squaring both sides, we obtain $(g_{n+2})^2 = 4g_{n+1}g_n$. By making the substitution

$$b_n = \log(g_n),$$

we obtain a nonhomogeneous linear recurrence, $2b_{n+2} = \log 4 + b_{n+1} + b_n$, whose solution is computed here, using a method which is analogous to that of solving a similar differential equation (such as $f(x) = 17 + f'(x) + f''(x)$). To begin, we identify the characteristic equation as $q(x) = 2x^2 - x - 1 = (2x+1)(x-1)$, which has roots $x = -\frac{1}{2}$ and $x = 1$. Thus, the homogeneous solution is $b_n = c_1(-\frac{1}{2})^n + c_2(1)^n$. To obtain the particular solution, we will apply the method of undetermined coefficients, that is, making an educated guess for the form of $f(n)$ in $b_n = f(n)\log(4)$. We use the initial conditions b_4 and b_5 from the recurrence to verify our guess that $b_n = An\log(4)$. Then, $2An\log(4) - A(n-1)\log(4) - A(n-2)\log(4) = \log(4)$, so $A = 1/3$. Thus, $b_n = \frac{n}{3}\log(4) + c_1(-\frac{1}{2})^n + c_2(1)^n$. By substituting b_4 and b_5 as initial conditions, we can solve for c_1 and c_2. Hence, we now construct and solve the following system of equations:

$$b_4 = \tfrac{3}{2}\log(2) = \tfrac{4}{3}\log(4) + \tfrac{1}{16}c_1 + c_2$$
$$b_5 = \tfrac{9}{4}\log(2) = \tfrac{5}{3}\log(4) - \tfrac{1}{32}c_1 + c_2$$

Solving for c_1 and c_2 yields $c_1 = -\frac{4}{9}\log(4)$ and $c_2 = -\frac{5}{9}\log(4)$. So, the solution to our recurrence relation is $b_n = \log(4)(\frac{n}{3} - \frac{4}{9}(-\frac{1}{2})^n - \frac{5}{9}(1)^n)$. Thus, for $n \geq 1$, we have the following closed-form expression for our geometric Fibonacci sequence:

$$g_n = \exp(b_n) = 2^{(\frac{2n}{3} - \frac{8}{9}(-\frac{1}{2})^n - \frac{10}{9})}.$$

As predicted by the simple calculation performed above, the asymptotic growth rate is indeed the cube root of four: $R_{gr} = \lim_{n\to\infty}(g_{n+1}/g_n) = 4^{1/3} = 1.5874\ldots$. Note that this rate of growth is close to that of the arithmetic (that is, the usual) Fibonacci sequence which we noted above as being the golden ratio, 1.6180..., but indeed less than the golden ratio, satisfying the geometric mean $\leq$ arithmetic mean inequality described above. Of course, however, just as we know that in the long-term, slight differences in interest rates result in large differences in bank account balances, for the same reason, the small difference in the growth rate with time results in quite large differences between the terms of the regular Fibonacci sequence and those of our geometric Fibonacci sequence.

Another way we can obtain this intriguing result is by examining the terms of $\{g_n\}$. Let us begin by writing a few terms of the sequence as powers of 2: $g_4 = 2^{3/2}, g_5 = 2^{9/4}, g_6 = 2^{23/8}, g_7 = 2^{57/16}$. If we write $g_n = 2^{m_n/2^{n-3}}$, we note that the terms in the numerator of the exponent (which we denote $\{m_n\}$), namely *3, 9, 23, 57,…*, (for $n = 4, 5, \ldots$) satisfy the recurrence relation $m_{n+2} = m_{n+1} + 2m_n + 2^{n-1}$. We can solve this recurrence relation in the same manner as that used to solve the recurrence of $\{g_n\}$. The characteristic equation of $m_{n+2} - m_{n+1} - 2m_n = 2^{n-1}$ is $(x^2 - x - 2) =$

$(x-2)(x+1)$, with solutions of $x = 2$ and $x = -1$. So, our homogeneous solution is $m_n = c_1(2)^n + c_2(-1)^n$. To obtain the particular solution, we again apply the method of undetermined coefficients. Suppose that $m_n = An(2)^n$. Then $2^{n-3} = An(2)^n - A(n-1)2^{n-1} - 2A(n-2)2^{n-2}$, so $A = \frac{1}{12}$. Thus, we have $m_n = c_1(2)^n + c_2(-1)^n + \frac{n}{12}(2)^n$. When we solve for c_1 and c_2, we obtain $c_1 = -\frac{5}{36}$ and $c_2 = -\frac{1}{9}$, which gives the following closed-form expression for $n \geq 4$:

$$m_n = -\tfrac{5}{36}(2)^n + -\tfrac{1}{9}(-1)^n + \tfrac{n}{12}(2)^n.$$

When using the relation between g_n and m_n, namely that $g_n = 2^{m_n/2^{n-3}}$, we obtain the same expression for g_n as the one we obtained with the previous method.

Finally, we analyze our *harmonic* Fibonacci sequence $\{h_n\}$, whose recurrence relation we recall is given by $h_{n+2} = \frac{4}{\frac{1}{h_n} + \frac{1}{h_{n+1}}}$. Again, it is not intuitively clear what type of growth this sequence undergoes, but we find that it too experiences exponential growth. By employing a heuristic procedure similar to that of the classical Fibonacci derivation, here we determine the limiting ratio $R_h = \lim_{n\to\infty} \frac{h_{n+1}}{h_n}$. Rearranging the recurrence relation yields $h_{n+2}h_{n+1} + h_{n+2}h_n = 4h_nh_{n+1}$. Manipulating the harmonic expression further yields $\frac{h_{n+2}}{h_n} + \frac{h_{n+2}}{h_{n+1}} = 4$. Thus, $R_h^2 + R_h = 4$, and by the quadratic formula, we obtain roots $\frac{-1\pm\sqrt{17}}{2}$. Finally, our growth rate is known to be positive, so $R_h = \frac{-1+\sqrt{17}}{2}$=1.5615....

Another way we can prove this is by the second method presented for the calculation of the growth rate of the geometric fibonacci sequence. Notice that each of terms of h_n for $n \geq 4$ are of the form $2^{2n-5}/j_n$, where $j_4 = 3, j_5 = 7$ and $j_{n+2} = j_{n+1} + 4j_n$ for $n \geq 6$. We can solve this recurrence relation by the methods described above, which gives the following closed-form expression for $n \geq 4$:

$$j_n = \frac{51+5\sqrt{17}}{1088}\left(\frac{1+\sqrt{17}}{2}\right)^n + \frac{51-5\sqrt{17}}{1088}\left(\frac{1-\sqrt{17}}{2}\right)^n.$$

When using the relation between h_n and j_n, namely that $h_n = \frac{2^{2n-5}}{j_n}$, we obtain an explicit expression for h_n. This gives us an asymptotic growth rate of $\frac{4}{(1+\sqrt{17})/2} = \frac{-1+\sqrt{17}}{2}$, as desired.

Thus, we have constructed the arithmetic-geometric-harmonic inequality for the growth rates:

$$\frac{1+\sqrt{5}}{2} \geq 4^{\frac{1}{3}} \geq \frac{-1+\sqrt{17}}{2},$$

with corresponding decimal approximations:

$$1.6180\ldots \geq 1.5874\ldots \geq 1.5615\ldots,$$

where the three terms correspond to the asymptotic growth rates we determined for the arithmetic (i.e., the usual), geometric, and harmonic Fibonacci sequences.

Possible Appendix: Integer-Valued Versions of the Geometric and Harmonic Fibonacci Sequences

It is interesting to note that although the growth rate of the Fibonacci sequence is an irrational number, namely the golden ratio, each term of the sequence is an integer. Note, however, that neither the geometric nor harmonic Fibonacci sequence is a sequence of integers. So we now define sequences whose recurrences are given by rounding up to the nearest integer twice the geometric or harmonic mean of the previous two terms; that is, consider, for example, a *rounded up* version of the geometric Fibonacci sequence, which we denote $\{g_n^u\}$:

$$g_{n+2}^u = \lceil 2\sqrt{g_{n+1}^u g_n^u} \rceil \text{ with } g_1^u = g_2^u = 1.$$

By bounding this sequence above and below, we can show that it has the same growth rate as that of the regular geometric Fibonacci sequence $\{g_n\}$. Similarly, a rounded down version of $\{g_n\}$ or a rounded up or rounded down version of the harmonic Fibonacci sequence $\{h_n\}$ can be shown to have the same growth rates as the corresponding non-rounded versions. The calculation for the growth rate of $\{g_n^u\}$ is performed here. Note, in addition, that it is initially unclear whether rounded down versions of these sequences are even increasing. For example, consider the sequence given by the recurrence $d_{n+2} = 2.5d_{n+1} - d_n$, with $d_1 = 20$, $d_2 = 10$. While this sequence approaches zero, in fact, the corresponding rounded down version is decreasing for all $n \geq 1$ (20, 10, 5, 2, 0, -2, -5, -11, $\ldots$) and negative for $n > 5$. The absolute value of the terms of this sequence grows exponentially. When we consider the rounded-up version, we see that for $n \geq 6$, the nth term is $(20/256)2^n$. (The first few terms of this sequence are 20, 10, 5, 3, 3, 5, 10, 20, 40, 80, $\ldots$.) From this example, we see that rounded up and rounded down sequences may differ vastly from the original sequence. The above example is adopted from one mentioned by past NCTM President Johnny Lott in a recent plenary address to the Tennessee Math Teachers Association in Memphis. See [3] for a comprehensive theory of rounding.

Now, we verify that the growth rate of the rounded up version of the geometric Fibonacci sequence $\{g_n^u\}$ given by $g_{n+2}^u = \lceil 2\sqrt{g_{n+1}^u g_n^u} \rceil$ is the same as that of the usual geometric Fibonacci sequence $\{g_n\}$. We bound the sequence above and below by sequences whose growth rate is the same as that of $\{g_n\}$. We define sequences:

$$u_{n+2} = 2\sqrt{u_{n+1}u_n} + 1, \text{with } u_1 = u_2 = 1,$$

$$d_{n+2} = \begin{cases} f_n & \text{for } n \leq 9 \\ 2\sqrt{d_{n+1}d_n} - 1 & \text{for } n > 9, \end{cases}$$

where $\{f_n\}$ denotes the usual Fibonacci sequence. It is interesting to note that the first nine terms of $\{g_n^u\}$ coincide precisely with those of $\{f_n\}$, the usual Fibonacci sequence. However, this simply illustrates that for pairs of small numbers, the corresponding arithmetic and geometric means are close. It is clear that the following inequalities hold:

$$d_n \leq g_n^u \leq u_n.$$

So it suffices to show that the growth rates of $\{u_n\}$ and $\{d_n\}$ are $4^{\frac{1}{3}}$, which is the growth rate of the regular geometric Fibonacci sequence $\{g_n\}$. Note that $\{u_n\}$ is clearly an increasing sequence for $n \geq 2$. Thus we have:

$$u_{n+2} = 2\sqrt{u_{n+1}u_n} + 1$$

$$\Rightarrow U := \lim_{n\to\infty} \tfrac{u_{n+2}}{u_{n+1}} = 2\lim_{n\to\infty} \tfrac{\sqrt{u_n}}{\sqrt{u_{n+1}}} + \lim_{n\to\infty} \tfrac{1}{u_{n+1}}$$

$$\Rightarrow U = 4^{\frac{1}{3}}.$$

The reason the definition of $\{d_n\}$ gives the first few terms is to ensure that the sequence is non-decreasing. Thus, the same argument implies that $\{d_n\}$ has the same rate of growth.

Similarly, the rounded down version of $\{g_n\}$ or the rounded up or rounded down version of the harmonic Fibonacci sequence $\{h_n\}$ can be shown to have the same growth rate as that of the corresponding non-rounded version; again we have to be careful to make sure the sequences corresponding to $\{d_n\}$ are actually non-decreasing, meaning that we may have to provide the first few terms, but that thereafter the recurrence, and thus the desired rate of growth, hold.

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