Domination Cover Pebbling: Structural Results

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Abstract

This paper continues the results of "Domination Cover Pebbling: Graph Families." An almost sharp bound for the domination cover pebbling (DCP) number, $\psi(G)$, for graphs G with specified diameter has been computed. For graphs of diameter two, a bound for the ratio between $\lambda(G)$, the cover pebbling number of G, and $\psi(G)$ has been computed. A variant of domination cover pebbling, called vertex neighbor integrity DCP is introduced, and preliminary results are discussed.

1 Introduction

Given a graph G we distribute a finite number of indistinguishable markers called *pebbles* on its vertices. Such an arrangement of pebbles, which can also be thought of as a function from $V(G)$ to $\mathbb{N} \cup \{0\}$, is called a *configuration.* A *pebbling move* on a graph is defined as taking two pebbles off one vertex, throwing one away, and moving the other to an adjacent vertex. Most research in pebbling has focused on a quantity known as the *pebbling number* $\pi(G)$ of a graph, introduced by F. Chung in [2], which is defined to be the smallest integer n such that for every configuration of n pebbles on the graph and for any vertex $v \in G$, there exists a sequence of pebbling moves starting at this configuration and ending in a configuration in which there is at least one pebble on v. A new variant of this concept, introduced in by Crull et al. in [6], is the *cover pebbling number* $\lambda(G)$, defined as the minimum number m such that for any initial configuration of at least m pebbles on G it is possible to make a sequence of pebbling moves after which there is at least one pebble on every vertex of G.

In a recent paper ([7]) the authors, along with Gardner, Godbole, Teguia, and Voung, have introduced a concept called domination cover pebbling and have presented some preliminary results. Given a graph G, and a configuration c, we call a vertex $v \in G$ *dominated* if it is covered (occupied by a pebble) or adjacent to a covered vertex. We call a configuration c' *domination cover pebbling solvable*, or simply *solvable*, if there is a sequence of pebbling moves starting at c' after which every vertex of G is dominated. We define the *domination cover pebbling number* $\psi(G)$ to be the minimum number n such that any initial configuration of n pebbles on G is domination cover pebbling solvable.

The set of covered vertices in the final configuration depends, in general, on the initial configuration—in particular, S need not equal a minimum dominating set. For instance, consider the configurations of pebbles on P_4, the path on four vertices, as shown in Figure 1:

Figure 1: An example where two different initial configurations produce two different domination cover solutions.

For the graph on the left, we make pebbling moves so that the first and third vertices (from left to right) form the vertices of the dominating set. However, for the graph on the right, we make pebbling moves so that the second and fourth vertices are selected to be the vertices of the dominating set. In some cases, moreover, it takes more vertices than are in the minimum dominating set of vertices to form the domination cover solution. For example, in Figure 2 we consider the case of the binary tree with height two, where the minimum dominating set has two vertices, but the minimal dominating set possible for a domination cover solution has three vertices. This corresponds to several possible starting configurations, for example the configuration pictured, the configuration with a pebble at the leftmost bottom vertex

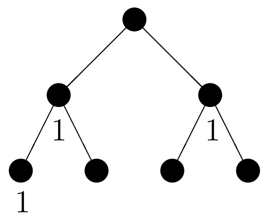

Figure 2: A reachable minimal configuration of pebbles on B_2 that forces a domination cover solution.

and 4 pebbles at the root, and the configuration with 1 and 10 pebbles at the leftmost and rightmost bottom level vertices respectively.

The above two facts constitute the main reason why domination cover pebbling is nontrivial. We refer the reader to [8] for additional exposition on domination in graphs, and to [7] for some further explanation of the domination cover pebbling number, including the computation of the domination cover pebbling number for some families of graphs.

One way to understand the size of the numbers $\pi(G), \lambda(G)$, and $\psi(G)$ is to find a bound for the size of these numbers given the diameter of G and the number of vertices. This has been done for $\pi(G)$ for graphs of diameter two in [5] and for graphs of diameter three in [1]. A theorem proven in [9] and [10] gives as a corollary a sharp bound for graphs of all diameters, which was originally established by other means in [11]. In this paper, we prove that for graphs of diameter two with n vertices, $\psi(G) \leq n - 1$. For graphs of diameter d, we show $\psi(G) \leq 2^{d-2}(n-2) + 1$. We also compute that the ratio $\lambda(G)/\psi(G) \geq 3$ for graphs of diameter two.

Another way to extend cover pebbling is called vertex neighbor integrity domination cover pebbling. A parameter ω used in calculating the vertex neighbor integrity of a graph G counts the size of the largest undominated connected subset of G. When $\omega = 0$, this corresponds to domination cover pebbling. To conclude this paper, we provide some preliminary results for this generalized parameter.

2 Diameter Two Graphs

In the next few sections, we will present structural domination cover pebbling results.

Theorem 2.1. *For all graphs G of order n with maximum diameter two, $\psi(G) \leq n - 1$.*

Proof. First, we show this bound is sharp by exhibiting a graph G such that $\psi(G) > n-2$. Consider the star graph on n vertices, and place a pebble on all of the outer vertices except one. This configuration of pebbles does not dominate the last outer vertex. Hence, $\psi(G) > n-2$.

To prove the theorem, we will show that, given a graph G of diameter two on n vertices, any configuration c of $n-1$ pebbles on G is solvable.

Given such a graph configuration c, let S_1 be the set of vertices $v \in G$ such that $c(v) > 1$. Let S_2 be the set vertices $w \in G$ such that $c(w) = 0$ and w is adjacent to some vertex of S_1, and let S_3 be the rest of the vertices, the ones that are neither in S_1 nor adjacent to a vertex of S_1. Let $a := |S_2|$, and $b := |S_3|$. Given a configuration c', define the *pairing number* $P(c')$ to be $\sum_{v \in G} \max\{0, \frac{c'(v)-1}{2}\}$. It can easily be checked that $P(c') = \frac{a+b-1}{2}$. Note that if $P(c') = k$ then c' contains at least $\lceil k \rceil$ disjoint pairs of pebbles, which means that we can make at least $\lceil k \rceil$ pebbling moves. Also, note that every vertex in G is at distance at most two from some vertex in S_1. This ensures that that every vertex in S_3 is adjacent to a vertex in S_2. Also, if some vertex in S_1 is not adjacent to a vertex of S_2, it must be adjacent only to vertices in S_1. Since this vertex has distance at most two from any other vertex on the graph, we conclude that every vertex of the graph is either in S_1 or adjacent to a vertex of S_1, meaning the G is already dominated by covered vertices, as desired. Therefore, it suffices to consider the case in which S_2 is a dominating set of G.

First, suppose that $a \leq b$. In this case, $P(c) \geq \frac{2a-1}{2}$. Hence, there are at least a disjoint pairs of pebbles that can be moved from elements in S_1 to S_2. For each uncovered vertex $v \in S_2$, if possible, move a pair of pebbles from an adjacent element of S_1 to put a pebble on v. After this is done for as many vertices of S_2 as possible, let L be the set vertices in S_2 which are still uncovered. Note that these vertices are necessarily at distance 2 from all remaining pairs of pebbles. Furthermore, since S_1 initially had at least a disjoint pairs of pebbles, there remain at least as many pairs as there are vertices in L. If this number is 0, the dominating set S_2 is covered and we are done. Otherwise, we nonetheless now know S_3 is dominated because if there were some vertex y that were adjacent to only those elements of S_2 which are also in L, then the minimum distance between y and a vertex in S_1 with a pair of pebbles is 3, which is impossible. However, it may be the case for some $z \in L$ that the vertex in S_1 that z was adjacent to lost its pebbles, and if this is the case, move a pair of pebbles from S_1 so that z is dominated (this always possible since our graph has diameter two). With the $|L|$ pairs we of pebbles we have, we can ensure each vertex of L is dominated. After this is done, G will be completely dominated by covered vertices.

Now consider the case $a > b$. We know that $P(c) \geq \frac{2b-1}{2}$ and so there are at least b pairs of pebbles available. Given any vertex v in S_3 and a pair of pebbles on a vertex $w \in S_1$, we can use this pair to move to a vertex between v and w, which is clearly in S_2. We now do this whenever necessary for each vertex of S_3, first using those pairs which can be removed from vertices having at least 3 pebbles. Let m be the number of moves that have been made. Then we know that m vertices in S_2 now have pebbles on them. Furthermore we know $m \leq b$, and since some of our moves may dominate multiple vertices of S_3, thus making some other moves unnecessary, it is indeed possible that $m < b$. In any case, after the moves are made, every vertex in $S_3 \cup S_1$ is dominated. If every vertex we have removed pebbles from is still covered, then the vertices of S_2 are still dominated and we are done.

Otherwise, we have removed pebbles from some vertex which had exactly two pebbles on it. Thus, these first m pebbling moves subtract at most $\frac{2m-1}{2}$ from $P(c)$, leaving a pairing number of $\frac{a+b-2m}{2} \geq \frac{a-m}{2}$ for the configuration after these moves. At this point, since we were forced to use pebbles from a vertex that had only two pebbles, we know that every vertex that contributes to the pairing number has exactly two pebbles on it. Thus there are at least $a-m$ vertices in S_1 with two pebbles on them. We can use these pairs to dominate the $a-m$ vertices of S_2 which are not covered. This leaves G dominated by covered vertices and therefore $\psi(G) \leq n-1$. $\square$

We can apply this theorem to prove a result about the ratio between the cover pebbling number and the domination cover pebbling number of a graph. We conjecture that this ratio holds for all graphs, but it does not seem that this can be directly proven using the structural bounds in this paper.

Theorem 2.2. *For all graphs G of order n with diameter two, $\lambda(G)/\psi(G) \geq 3$.*

Proof. First, suppose that the minimum degree of a vertex of G is less than or equal to $\lceil \frac{n-1}{2} \rceil$. By the previous theorem, we know that the maximum value of $\psi(G)$ is $n-1$. We now construct a configuration of pebbles on G such that $\lambda(G) \geq 3n-3$. Place $3n-3$ pebbles on any vertex v that has a degree less than $\lceil \frac{n-1}{2} \rceil$. It takes 2 pebbles to cover solve each vertex adjacent to v, at most $\lceil \frac{n-1}{2} \rceil$, and all the remaining vertices require 4 pebbles. Since there are at least as many vertices a distance of 2 away from v as there are a distance of 1 away from G, $3n-3$ pebbles or more are required to cover pebble all of the vertices except for v. Thus for this class of graphs, $\lambda(G) > 3n-3 \geq 3\psi(G)$.

Now suppose that the minimum degree k of a vertex in G is greater than $\lceil \frac{n-1}{2} \rceil$. By a similar argument as the previous paragraph, notice that $\lambda(G)$ for any diameter two graph is at least $4n-2m-3$, where m is the minimum degree of a vertex of G. Since $\lambda(G) \geq 4n-2m-3$, it suffices to show we can always solve a configuration c

of $\lfloor \frac{4n-2m-3}{3} \rfloor = k$ pebbles on G. Given a particular value for m between $\lceil \frac{n+1}{2} \rceil$ and $n-1$, we will construct a domination cover solution.

As long as there exist vertices of G that have at least three pebbles and are adjacent to an unoccupied vertex, we haphazardly make moves from such vertices to adjacent unoccupied vertices. We claim that the resulting configuration has the desired property that the set of occupied vertices are a dominating set of G. First suppose that the algorithm is forced to terminate while there remains some vertex v having at least three pebbles. Then this vertex must be adjacent only to occupied vertices of G, and since the diameter of G is two, these neighbors v form a dominating set of G. Otherwise, if every vertex has less than three pebbles, it can easily be checked that the number of occupied vertices is now $\sum_{v \in G} \lceil \frac{c(v)}{2} \rceil \geq \lceil \frac{k}{2} \rceil$. Since the minimum degree of a vertex in G is m, by the pigeonhole principle, if we now have $n-m$ or more vertices covered by a pebble, then every vertex of G is dominated. So if $\lceil \frac{k}{2} \rceil \geq n-m$, we are finished. We see that

$$\left\lceil \frac{\lfloor \frac{4n-2m-3}{3} \rfloor}{2} \right\rceil \geq \left\lceil \frac{\frac{4n-2m-5}{3}}{2} \right\rceil = \left\lceil \frac{4n}{6} - \frac{m}{3} - \frac{5}{6} \right\rceil$$

Therefore, we are done if

$$\left\lceil \frac{4n}{6} - \frac{m}{3} - \frac{5}{6} \right\rceil \geq n - m,$$

which is equivalent to

$$n \leq \left\lceil \frac{4n}{6} + \frac{2m}{3} - \frac{5}{6} \right\rceil.$$

This inequality holds for $m \geq \lceil \frac{n+1}{2} \rceil$. Therefore, we have completed this case and have shown that for all graphs G of diameter two, $\lambda(G)/\psi(G) \geq 3$. □

We now prove a more general bound for graphs of diameter d.

3 Graphs of Diameter d

Theorem 3.1. *Let G be a graph of diameter $d \geq 3$ and order n. Then $\psi(G) \leq 2^{d-2}(n-2)+1$.*

Throughout the proof, we adopt the convention that if G is a graph and V and W are subsets of $V(G)$ and $v \in V(G)$ then $d(v, W) = \min_{w \in W} d(v, w)$ and $d(V, W) = \min_{v \in V} d(v, W)$. Also, for any set $S \subseteq V(G)$ we of course let $S^C = V(G) \setminus S$.

Proof. First, we define the *clumping number* χ of a configuration c' by

$$\chi(c') := \sum_{v \in G} 2^{d-2} \max\left(\left\lfloor \frac{c'(v)-1}{2^{d-2}} \right\rfloor, \ 0\right).$$

The clumping number counts the number of pebbles in a configuration which are part of disjoint "clumps" of size 2^{d-2} on a single vertex, with one pebble on each occupied vertex ignored.

Now let c be a configuration on G of size at least $2^{d-2}(n-2)+1$. We will show that c is solvable by giving a recursively defined algorithm for solving c through a sequence of pebbling moves. First, we make some definitions to begin the algorithm:

- $c_0 = c$.
- $A_0 = \{v \in G \ : \ c(v) > 0\}$.
- $B_0 = \{v \in G \ : \ c(v) \geq 2^{d-2}+1\}$.
- $C_0 = V(G) - A_0$.
- $D_0 = \emptyset$.

We will describe our algorithm by recursively defining a sequence of configurations c_p and four sequences A_p, B_p, C_p, and D_p of sets of vertices. At each step, we will need to make sure a few conditions hold, to ensure that the next step of the algorithm may be performed. For each m, we will insist that:

1. For every $v \in C_m \cup D_m$, $c_m(v) = 0$ and for every $v \in A_m$, $c_m(v) > 0$.
2. $\chi(c_m) \geq 2^{d-2}(|C_m| - 1)$.
3. $|C_m| \leq |C_0| - m$.
4. $B_m = \{v \in G \ : \ c_m(v) \geq 2^{d-2}+1\}$.
5. If both $B_m \neq \emptyset$ and $D_m \neq \emptyset$, $d(B_m, D_m) = d$; If $D_m \neq \emptyset$, there always exists some $v \in G$ such that $d(v, D_m) = d$, even if $B_m = \emptyset$.
6. A_m, C_m, and D_m are pairwise disjoint and $A_m \cup C_m \cup D_m = V(G)$.
7. Every vertex of D_m is dominated by c_m.
8. There exists a sequence of pebbling moves transforming c to c_m.

Note by 1, 4, and 6, we will always have $B_m \subseteq A_m$. Also, by 1, 6, and 7, every vertex of G which is not dominated by c_m is in C_m.

For $m = 0$, only condition 2 is not immediately clear. To verify it, note that

$$\begin{aligned}\chi(c) &= \sum_{v\in G} 2^{d-2} \max\left(\left\lfloor \frac{c(v)-1}{2^{d-2}} \right\rfloor,\ 0\right) \\ &= \sum_{v\in A_0} 2^{d-2} \left\lfloor \frac{c(v)-1}{2^{d-2}} \right\rfloor \\ &\geq \sum_{v\in A_0} 2^{d-2} \left(\frac{c(v)}{2^{d-2}} - 1\right).\end{aligned}$$

Using the fact that the size of c is at least $2^{d-2}(n-2)+1$, and $|C_0| = n - |A_0|$, we see

$$\chi(c) \geq (2^{d-2}(n-2)+1) - 2^{d-2}|A_0| = 2^{d-2}(|C_0|-2)+1.$$

From the definition of χ, it is apparent that $2^{d-2} | \chi(c)$. Thus, we indeed must have

$$\chi(c) = \chi(c_0) \geq 2^{d-2}(|C_0|-1).$$

Suppose for some $p-1 > 0$ we have defined $c_{p-1}, A_{p-1}, B_{p-1}, C_{p-1}$, and D_{p-1} and the above conditions hold when $m = p-1$. We shall assume that there is some vertex in C_{p-1} which is not dominated by c_{p-1}, for otherwise, by conditions 6, 7 and 8, c is solvable and we are done. Thus $|C_{p-1}| \geq 1$. But suppose $|C_{p-1}| = 1$. Call this single vertex v. Since it is non-dominated, it is adjacent to only uncovered vertices. These vertices cannot be in C_{p-1} for $|C_{p-1}| = 1$, and they are not in A_{p-1}, because every vertex in A_{p-1} is covered by property 1. So every vertex adjacent to v is in D_{p-1}. Invoke property 5 to choose a $w \in G$ for which $d(w, D_{p-1}) = d$. Any path from w to v passes through one of the vertices in D_{p-1} which is adjacent to v, and is thus of length at least $d+1$, so $d(w,v) \geq d+1$, contradicting the assumption that G has diameter d. We have now shown that, if C_{p-1} has a non-dominated vertex, then $|C_{p-1}| \geq 2$. In this case, we will have $\chi(c_{p-1}) \geq 2^{d-2}$, ensuring the existence of some clump of size 2^{d-2}, and thus that B_{p-1} is non-empty. Therefore, we will always implicitly assume that $B_{p-1} \neq \emptyset$.

Case 1: $d(B_{p-1}, C_{p-1}) \leq d-2$

In this case, we choose $v' \in B_{p-1}$ and $w' \in C_{p-1}$ for which $d(v', w') \leq d-2$ and move $2^{d(v',w')}$ pebbles from v' to w', leaving one pebble on w' and at least one on v'. We let c_p be the configuration of pebbles resulting from this move. Let $C_p = C_{p-1} \backslash w'$.

Thus $|C_p| = |C_{p-1}| - 1 \leq |C_0| - (p-1) - 1$ and we see that condition 3 holds when $m = p$. Furthermore, We have used at most one clump of 2^{d-2} pebbles so

$$\chi(c_p) \geq \chi(c_{p-1}) - 2^{d-2} \geq 2^{d-2}(|C_{p-1}| - 1) - 2^{d-2} = 2^{d-2}(|C_p| - 1)$$

and therefore condition 2 holds for p. Also, we let $A_p = A_{p-1} \cup \{w'\}$, let $C_p = C_{p-1}\ w'$, and $D_p = D_{p-1}$ (now, clearly condition 6 holds.) We again let $B_p = \{v \in G\ :\ c_p(v) \geq 2^{d-2} + 1\}$, which simply means that we have possible removed v' from B_{p-1} if v' now has less than $2^{d-2} + 1$ pebbles. Thus $B_p \subseteq B_{p-1}$, and now 1, 4, 5, 7, and, 8 are all easily seen to hold for $m = p$.

Case 2: $d(B_{p-1}, C_{p-1}) \geq d - 1$.

If every vertex in C_{p-1} is dominated by A_{p-1}, we are done. Otherwise, let w' be some non-dominated vertex in C_{p-1}. Clearly, w' is at distance $d-1$ or d from B_{p-1}. Suppose $d(B_{p-1}, w') = d - 1$. Then w' is adjacent to some (non-covered) vertex w'' at distance $d - 2$ from B_{p-1}. By condition 1, every vertex of G which is not covered by c_{p-1} is in $C_{p-1} \cup D_{p-1}$. But $d(B_{p-1}, C_{p-1}) \geq d - 1$ and by 5, $d(B_{p-1}, D_{p-1}) = d$ so $w'' \notin C_{p-1} \cup D_{p-1}$. This contradiction means that $d(w', B_{p-1}) \neq d - 1$ and so $d(w', B_{p-1}) = d$.

Choose some vertex in B_{p-1} and call it v'. We know $d(v', w') = d$ so consider some path of length d from v' to w'. Let v^* be the unique point on this path for which $d(v^*, v' = d - 2)$. Thus $v^* \notin C_{p-1} \cup D_{p-1}$ and so $v^* \in A_{p-1}$, and also $d(v^*, w') = 2$. Let w'' be some vertex which is adjacent to both v^* and w' so that $d(v', w'') = d - 1$. Then because w'' is uncovered (else w' would be dominated), it must be in C_{p-1}. This also means that $v^* \notin B_{p-1}$ by the assumption that $d(B_{p-1}, C_{n-1}) \geq d - 1$.

We now move one clump of 2^{d-2} pebbles from v' to v^*, adding one pebble to v^*, which now, by condition 1, has at least two pebbles. We then move two pebbles from v^* and cover w'' with one pebble. We let c_p be the configuration resulting from these moves. We let $D_p = D_{p-1} \cup \{w'\}$ and we again let $B_p = \{v \in G\ :\ c_p(v) \geq 2^{d-2} + 1\}$, which just means we have possibly removed v' from B_{p-1}, so $B_p \subseteq B_{p-1}$. If now $c_p(v^*) = 0$, we let $A_p = A_{p-1} \cup \{w''\} \setminus v^*\}$ and $C_p = C_{p-1} \cup \{v^*\} \setminus \{w', w''\}$. Otherwise, if $c_p(v^*) > 0$, let $A_p = A_{p-1} \cup \{w''\}$ and $C_p = C_{p-1} \setminus \{w', w''\}$. This ensures that conditions 1 and 6 still hold for $m = p$. Also, $|C_p| \leq |C_{p-1}| - 1 \leq |C_0| - (p-1) - 1$ and so condition 3 holds for $m = p$. Furthermore, we have used only one clump of 2^{d-2} pebbles, because $v^* \notin B_{p-1}$ and so by using a pebble from v^*, we could not have destroyed a clump. Thus

$$\chi(c_p) = \chi(c_{p-1}) - 2^{d-2} \geq 2^{d-2}(|C_{p-1}| - 1) - 2^{d-2} \geq 2^{d-2}(|C_p| - 1)$$

and therefore condition 2 holds for p. Condition 5 also still holds for $m = p$ because $B_p \subseteq B_{p-1}$ and because we have added only the vertex w' to D_{p-1} and $d(B_{p-1}, w') = d$, so $d(B_{p-1}, D_p) = d$. To see condition 7 is still true, note that to get D_p we have only added w' to D_{p-1}, and certainly, w' is adjacent to w'', which is covered by c_p, so w' is dominated by c_p. Also, the only previously covered vertex of G which is now uncovered is (possibly) v^* but $d(v^*, B_{p-1}) = d - 2$, and so v^* is not adjacent to any vertex in D_{p-1} for, by 5, $d(B_{p-1}, D_{p-1}) = d$. Thus, by possibly uncovering v^*, we did not cause any vertex in D_{p-1} to become undominated, so 7 still holds for $m = p$. Finally, the fact that conditions 4 and 8 still hold for $m = p$ is easily seen.

The algorithm continues as long as there is some non-dominated vertex in C_p. By condition 3, it must terminate after at most $|C_0|$ steps, with $|C_k| = 0$ for some $k \leq |C_0|$. The configuration c_k clearly dominates every vertex of G, and by property 8, c_k is reachable from c by pebbling moves, so c is solvable. □

For $d \geq 3$, Figure 3 shows a graph G which is an example of a graph of diameter d with $n = 2m + d - 2$ vertices for which $\psi(G)$ comes close to the upper bound of $2^{d-2}(n-2) + 1 = 2^{d-1}m + 2^{d-2}(d-2) + 1$.

To dominate vertex w_i, it is easy to see a pebble is needed on w_i or v_i. They each have distance not less than $d - 1$ from u_{d-1}, and so it requires 2^{d-1} pebbles on u_{d-1} to supply this pebble. This means at least $2^{d-1}m$ pebbles are needed on u_{d-1} to dominate every w_i, so $\psi(G) \geq 2^{d-1}m$. Further, using the result of [9] and [10], we can calculate $\lambda(G) = 3 \cdot 2^{d-1}m + 2^d - 1$. Clearly, by making m large we can make $\lambda(G)/\psi(G)$ arbitrarily close to 3. Also note that for the complete graph on 2 vertices, $\lambda(G) = 3$ and $\psi(G) = 1$. We conjecture that it is not possible, however, for the ratio to be less than 3:

Conjecture 3.1. $\lambda(G)/\psi(G) \geq 3$ *for all graphs* G *with more than one vertex.*

4 Vertex Neighbor Integrity DCP

Cozzens and Wu [4] created a graph parameter called the *vertex neighbor integrity*, or VNI, which has been the subject of numerous studies. We proceed to describe this parameter with the definitions of Cozzens and Wu [4]. Let $G = (V, E)$ be a graph and u be a vertex of G. The *open neighborhood* of u is $N(u) = \{v \in V(G) | \{u, v\} \in E(G)\}$; the *closed neighborhood* of u is $N[u] = \{u\} \cup N(u)$. Analogously, for any $S \subseteq V(G)$, define the *open neighborhood* $N(S) = \cup_{u \in S} N(u)$ and the *closed neighborhood* $N[S] = \cup_{u \in S} N[u]$. A vertex $u \in V(G)$ is *subverted* by removing the closed neighborhood $N[u]$

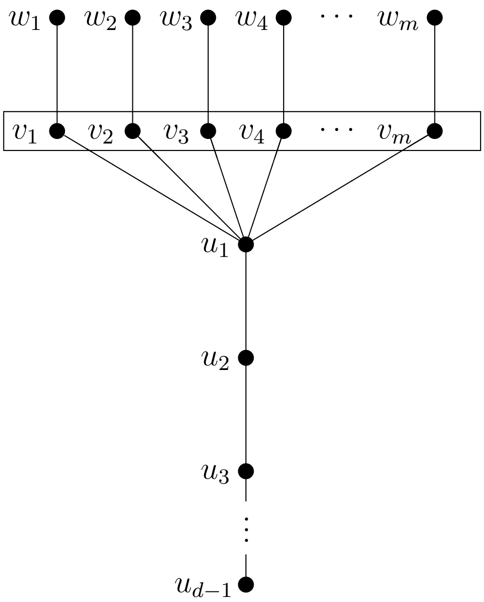

Figure 3: A graph with high DCP number. The box represents the fact that there is an edge between every pair of vertices inside, making the subgraph induced by $\{v_1, v_2, \ldots, v_m\}$ a complete graph on m vertices.

from G. Notice that this subversion is equivalent to the removal of a dominating set from G. For a set of vertices $S \subseteq V(G)$, the *vertex subversion strategy* S is applied by subverting each of the vertices of S from G. Define the *survival subgraph* to the the subgraph left after the subversion strategy is applied to G. The *order* of G is defined to be $|V(G)|$.

Definition 4.1. *The vertex neighbor integrity of a graph G is defined as*

$$VNI(G) = \min_{S \subseteq V(G)} \{|S| + \omega(G \setminus S)\},$$

where $w(H)$ is the order of the largest connected component in the graph H.

We apply a variant of subversion in order to describe how VNI calculations relate to domination cover solution. Let $\Omega_\omega(G)$ be the minimum number of pebbles required such that it is always possible to construct an incomplete domination cover pebbling of G, where disjoint undominated components of G can have order at most ω. This corresponds to the $\omega(G)$ term in the VNI computation. Notice that domination cover pebbling corresponds to the case when $\omega = 0$.

5 Basic Results

Theorem 5.1. *For $\omega \geq 0$, $\Omega_\omega(K_n) = 1$.*

Proof. When a pebble is placed on K_n, the entire graph is dominated. The result follows. $\square$

Theorem 5.2. *For $s_1 \geq s_2 \geq \cdots \geq s_r$, let $K_{s_1,s_2,\ldots,s_r}$ be the complete r-partite graph with $s_1, s_2, \ldots, s_r$ vertices in vertex classes $c_1, c_2, \ldots, c_r$ respectively. Then for $\omega \geq 1$, $\Omega_\omega(K_{s_1,s_2,\ldots,s_r}) = 1$.*

Proof. Place a pebble on any vertex in c_i. All the vertices in the other c_i's are dominated. The other vertices in c_1 that are undominated are disjoint from each other. Thus, the result follows. $\square$

Theorem 5.3. *For $\omega \geq 1$, $n \geq \omega + 3$, $\Omega_\omega(W_n) = n - 2 - \omega$, where W_n denotes the wheel graph on n vertices.*

Proof. First, we will show that $\Omega_\omega(W_n) > n - 3 - \omega$. Place a single pebble on each of $n - 3 - \omega$ consecutive outer vertices so that all of the pebbled vertices form a path. This leaves a connected undominated set of size $\omega + 1$. Hence, $\Omega_\omega(W_n) > n - 3 - \omega$. Now, suppose that we place $n - 2 - \omega$ pebbles on W_n. If any vertices have a pair of pebbles on them, the entire graph can be dominated by moving a single pebble to the hub vertex. Hence, each vertex can contain only one pebble. Since every outer vertex is of degree 3, if any vertex is undominated, at least 3 vertices must be dominated but unpebbled. Hence, in order to obtain an undominated set of size $\omega + 1$, there must be $\omega + 4$ vertices that are unpebbled. By the pigeonhole principle, we obtain a contradiction because there are not enough vertices for this constraint to hold. Thus, for $\omega \geq 1$, $n \geq \omega + 3$, $\Omega_\omega(W_n) = n - 2 - \omega$. $\square$

6 Graphs of Diameter 2 and 3

Theorem 6.1. *Let G be a graph of diameter two with n vertices. For $\omega \geq 1$, $\Omega_\omega(G) \leq n - 1 - \omega$.*

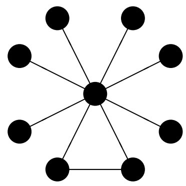

Figure 4: An example of the construction for $n = 9$, $\omega = 1$.

Proof. To show that the bound is sharp, consider the graph H_n, defined to be a star graph with n vertices with w additional edges added to make the graph induced by one subset of $w + 1$ outer vertices connected.

If we place a single pebble on each of the $n - 2 - \omega$ tendrils of the star that are not connected to any other tendrils, the remaining set of undominated vertices is connected and of size $\omega + 1$. Hence, $\Omega(H_n) > n - 2 - \omega$.

Now, let G be a graph with n vertices. Suppose there is an arbitrary configuration of pebbles $c(G)$ that contains exactly $n - 1 - \omega$ pebbles. To prove the theorem, we now show that a domination cover solution of G can be constructed such that the maximum order of an undominated component of G is ω. Using the algorithm presented in this proof, we can also prove the stronger statement that there are at most ω vertices that are undominated.

Let A be the set of all vertices $a \in G$ such that vertex a contains a single pebble. Let B be the set of vertices $b \in G$ such that vertex b contains two or more pebbles. Let C be the set of vertices of all $c \in G$ such that c is dominated but contains no pebbles. Let D be the set of all vertices in G such that if $d \in G$, then vertex d is undominated. Thus, all vertices in D are a distance of 2 from every element of $A \cup B$.

We now describe a process that forces $n - \omega$ vertices to be dominated. Let F be the set of vertices that are forced to be dominated and will remain dominated throughout the process. Since we never move pebbles from vertices with a single pebble on them, we have forced all of the vertices in A to be dominated. Thus for all $a \in A$, $a \in F$. If D is empty, then we have dominated the entire graph and the proof is complete. So suppose there exists some vertex v that is in D. Since G has a diameter of 2, then v can be dominated by moving a pair of pebbles from any vertex in B that still has at least 2 pebbles on them.

For every vertex v dominated in such a manner, two vertices become elements of F, namely v, and the empty vertex that the pair of pebbles moved to in order to dominate v. Perform this process repeatedly until the entire graph is dominated or there is only one vertex v^*, that has exactly 2 or 3 pebbles left on it and no other vertices have contain more than one pebble. Notice that the only vertices in F that are unpebbled are those that are a distance of two from every pair of vertices. Except for v^*, the vertices in B now either have zero or one pebble on it. If a vertex in B has one pebble on it, then that vertex also gets put into F. So far, for every pebble of the initial configuration of G except for the ones remaining on v^*, one pebble has forced at least one vertex to be in F.

First, consider the case where v^* has two pebbles. If there are $n - \omega$ vertices already in L, we are then finished because the maximum number of undominated vertices left is ω. Also notice that the only unpebbled vertices in F are those that are a distance of two away from the set of all pairs. Since the graph is undominated, there exists some vertex, d' in D that is undominated. In this case, moving the last pair of pebbles to dominate a vertex means that we have forced 3 verties not in F to be dominated, namely d', v^* and a vertex not already in F connecting them. Thus, since we have dominated at least $n - \omega$ total vertices, one vertex for each pebble plus an additional vertex, the largest undominated set possible is of size ω, and this case is complete.

If v^* has 3 pebbles and there is only one undominated vertex left, then moving a pair of pebbles to dominate v^* dominates the entire graph. Otherwise, there are at least two vertices that are undominated. If there is some common unpebbled vertex, x, that would dominate at least two undominated vertices, then using the last pair of pebbles to move a pebble to x will force at least 4 vertices to be dominated that are not members of F. These vertices are v^*, x, and two undominated vertices a distance of two away from v^*. Thus, after this operation, at least $n - \omega$ vertices are dominated. If there is no common unpebbled vertex, then there are at least two unpebbled vertices of distance 1 from v^* that have not been placed in F. Notice that the only unpebbled vertices that have been forced are those that are a distance of 2 away from the set of all pairs. So take the pair of pebbles and place a pebble on a vertex that forces two more vertices to be placed in F. The remaining pebble on v^* will force v^* and at least one more vertex adjacent to v^* that is empty and has not been forced to be dominated. Again, at least $n - \omega$ vertices are dominated, whence there cannot exist an undominated component of G that contains $\omega + 1$ or more vertices, and the proof is complete. $\square$

We conclude this section by conjecturing an analogous result for graphs of diameter 3, along with a valid lower-bound construction for this conjecture.

Conjecture 6.1. *Let G be a graph of diameter 3 with n vertices. For $i \geq 1$, $\Omega(G) \leq \lfloor \frac{3}{2}(n - 2 - \omega) + 1 \rfloor$.*

To see that this result is reasonable, we will show that $\Omega_i(G) > \lfloor \frac{3}{2}(n-2-\omega) \rfloor$. Consider the following family of graphs that are of size $\lfloor \frac{3}{2}(n-3-\omega)+1 \rfloor$ but contain an undominated component of order $\omega+1$. Take a $K_{\omega+1}$ and attach each of the ω vertices to some other vertex v. Connect v to each vertex of a $K_{\lceil \frac{n-\omega-2}{2} \rceil}$, call it H. Connect each of the remaining $\lfloor \frac{n-\omega-2}{2} \rfloor$ vertices to a vertex of H, so that each vertex in H has only one tendril off it. Now, place three pebbles on each of the tendril vertices, and if there is one vertex in H without a tendril, place one pebble on it. This is a total of $3\lfloor \frac{n-\omega-2}{2} \rfloor$ (+1 if $n-\omega-2$ is odd) pebbles in this configuration, which is equivalent to $\lfloor \frac{3}{2}(n-2-\omega) \rfloor$. Since it is not possible to dominate the vertices in $K_{\omega+1}$, the graph still has an undominated component of order $\omega+1$. Thus, $\Omega_i(G) > \lfloor \frac{3}{2}(n-2-\omega) \rfloor$.

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