# Cover Pebbling Numbers and Bounds for Certain Families of Graphs

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**Abstract**

Given a configuration of pebbles on the vertices of a graph, a *pebbling move* is defined by removing two pebbles from some vertex and placing one pebble on an adjacent vertex. The cover pebbling number of a graph, $\gamma(G)$, is the smallest number of pebbles such that through a sequence of pebbling moves, a pebble can eventually be placed on every vertex simultaneously, no matter how the pebbles are initially distributed. The cover pebbling number for complete multipartite graphs and wheel graphs is determined. We also prove a sharp bound for $\gamma(G)$ given the diameter and number of vertices of $G$. [1]

**Keywords:** graph, pebbling, diameter, coverable

## 1 Introduction

One recent development in graph theory, suggested by Lagarias and Saks, called pebbling, has been the subject of much research and substantive generalizations. It was first introduced into the literature by Chung [1], and has been developed by many others including Hurlbert, who published a survey of pebbling results in [3]. Given a connected graph $G$, distribute $k$ pebbles on its vertices in some configuration, $C$. Specifically, a *configuration*

[1] The results of this paper have recently been superseded by a theorem proven in [5] and [6], which gives a general formula for the cover pebbling number of graphs.

on a graph $G$ is a function from $V(G)$ to $\mathbb{N}\cup\{0\}$ representing an arrangement of pebbles on $G$. We call the total number of pebbles $k$ the *size* of the configuration. A *pebbling move* is defined as the simultaneous removal of two pebbles from some vertex and addition of one pebble on an adjacent vertex. A pebble can be moved to a root vertex $v$ if it is possible to place one pebble on $v$ in a sequence of pebbling moves. We define the pebbling number, $\pi(G)$ to be the minimum number of pebbles needed so that for any initial distribution of pebbles, it is possible to move to any root vertex $v$ in $G$.

The concept of cover-solvability was introduced in [2]. We call a configuration on a graph *cover-solvable* if, starting with this configuration, it is possible, through a sequence of pebbling moves, to simultaneously place one pebble on every vertex of the graph. The *cover pebbling number* of a graph, $\gamma(G)$, is defined as the smallest number such that every configuration of this size is cover-solvable. One application in [2] for $\gamma(G)$ is based on a military application in which troops must be distributed simultaneously.

In [2], the cover pebbling number for complete graphs, paths and trees is determined. Also, Hurlbert and Munyan [4] have constructed a proof that determines the cover pebbling number of the $d$-cube.

This paper will consider various questions related to cover pebbling, including open problem 9 in [2]. Section 2 describes the computation for calculating the cover pebbling number for complete multipartite graphs. In Section 3, we will compute the cover pebbling number of $W_n$, the wheel graph. We conclude the paper in Section 4 by constructing a tight upper bound for the cover pebbling number of graphs with specified diameter $d$ and number of vertices $n$.

# 2 Complete Multipartite Graphs

**Definition 2.1.** *For $s_1 \geq s_2 \geq \cdots \geq s_r$, let $K_{s_1,s_2,\ldots,s_r}$ be the complete $r$-partite graph with $s_1, s_2, \ldots, s_r$ vertices in vertex classes $c_1, c_2, \ldots, c_r$ respectively.*

**Definition 2.2.** *For a complete $r$-partite graph $G = K_{s_1,s_2,\ldots,s_r}$, let $\phi(G) = 4s_1 + 2s_2 + \cdots + 2s_r - 3$.*

**Theorem 2.3.** $\gamma(K_{s_1,s_2,\ldots,s_r}) = \phi(G)$.

*Proof.* First, we show that not every configuration of size $\phi(K_{s_1,s_2,\cdots,s_r}) - 1$ on $K_{s_1,s_2,\cdots,s_r}$ is cover-solvable. For instance, suppose all $\phi(K_{s_1,s_2,\cdots,s_r}) - 1$ pebbles are on one vertex of $c_1$, call it $x$. There are $k = s_2 + s_3 + \cdots + s_r$ vertices that are distance 1 from $x$ and $l = s_1 - 1$ vertices that are distance 2 from $x$. For the $k$ vertices a distance 1 from $x$, $2k$ pebbles are required to cover these vertices, and for the $l$ vertices at distance 2 from $x$, there are $4l$ pebbles required to cover these vertices. We need one more pebble to remain on $x$, for a total of $2k + 4l + 1 = \phi(K_{s_1,s_2,\cdots,s_r})$ pebbles required, which is one more than we have. Thus, this configuration is not cover-solvable.

Now suppose that there exists some complete $r$-partite graph $K_{s_1,s_2,\ldots,s_r}$ which has a configuration of size $\phi(K_{s_1,s_2,\ldots,s_r})$ that is not cover-solvable. Among such graphs, choose one of minimal order (let it be $G' = K_{s'_1,s'_2,\ldots,s'_{r'}}$).

First, we will show that $G'$ cannot be a star graph (that is, a $K_{s'_1,1}$). To see this, consider:

**Definition 2.4** (Crull et al [2]). *Let $T$ be a tree and let $V(T)$ be the vertex set of $T$. For $v \in V(T)$, define*

$$s(v) = \sum_{u \in V(T)} 2^{d(u,v)},$$

*with $d(u, v)$ denoting the distance from $u$ to $v$, and let*

$$s(T) = \max_{v \in V(T)} s(v).$$

In [2] it is shown that for any tree $T$, $\gamma(T) = s(T)$. Since $G'$ is a tree, we can compute $\gamma(G')$ by evaluating $s(v)$ for all $v \in G$ to obtain $s(G)$. If $v \in c_1$ then $s(v) = 4s'_1 - 1$, and if $v \in c_2$ then $s(v) = 2s'_1 + 1$. Thus, $s(G') = \gamma(G') = 4s'_1 - 1 = 4s'_1 + 2s'_2 - 3 = \phi(G')$. Hence, for a star, every configuration of size $\phi(G')$ is cover-solvable. Since $G'$ is not a star, further suppose that for any $G'$, each complete multipartite subgraph $G$ of $G'$ is cover-solvable with $\phi(G)$ pebbles.

Notice that for any complete $p$-partite graph with $p \geq 2$ other than a star graph, the removal of a vertex from the graph leaves a subgraph that is a complete $q$-partite graph with $q \geq 2$. Since $G'$ cannot be a star, for any vertex $v \in G'$, $G' - v$ is a complete $r^*$-partite graph with $r^* \geq 2$. Furthermore, since by our assumption of the minimality of $G'$, for any complete $r$-partite graph $G$ smaller than $G'$, a configuration of size $\phi(G)$ or greater must be cover-solvable, and since clearly $\phi(G' - v) \leq \phi(G') - 2$, any configuration of size $\phi(G') - 2$ or greater on $G' - v$ is cover-solvable.

Let $C$ be a configuration of size $\phi(G')$ on $G'$. Suppose $C(v) = 1$ or $2$ for some $v \in G'$. Then $C$ restricted to $G' - v$ is a configuration of size at least $\phi(G') - 2$ and thus is cover-solvable on $G' - v$. After we carry out the steps of the cover-solution of this subgraph, we will have cover-solved $G'$, contradicting our hypothesis.

Otherwise, if $C(v) = 0$ or $C(v) \geq 3$ for all $v \in G'$, choose some $v'$ for which $C(v') = 0$ (if no such $v'$ exists, we are done). Then consider the vertices of $G'$ which are in different vertex classes of $G'$ from $v'$. If at least one of these is initially occupied, call it $v''$. Then since $C(v'') \geq 3$, we can cover $v'$ with pebbles from $v''$, while leaving $\phi(G') - 2$ pebbles on $G' - v'$. Thus, the configuration of pebbles on $G'$ after this move, restricted to the subgraph $G' - v'$ is cover-solvable, and after we carry out the steps of the cover-solution of this subgraph, we will have cover-solved $G'$. Otherwise, all the vertices in the vertex classes of $G'$ that are different than the vertex class from $v'$ are empty. Thus, all pebbles are on vertices in the vertex class of $v'$, and in particular, some vertex $w$ of this class has pebbles on it, so $C(w) \geq 2$. Thus, we can use pebbles on $w$ to cover some vertex $w'$ in another vertex class, as all these vertices are empty. Note that after this move, the configuration of pebbles on $G' - w'$ has size $\phi(G') - 2$, and thus this configuration restricted to the subgraph $G' - w'$ is cover-solvable. Again, after we carry out the steps of the cover-solution of this subgraph, we will have cover-solved $G'$. □

# 3 The Wheel Graph

In this section, we will compute $\gamma(W_n)$, with $W_n$ denoting the wheel graph. The wheel graph is composed of a cycle consisting of $n$ vertices, $v_1, \ldots, v_n$, which are all connected to a hub vertex, $v_0$, for a total of $v = n+1$ vertices.

**Theorem 3.1.** *For $n \geq 3$, $\gamma(W_n) = 4n - 5 = 4v - 9$.*

*Proof.* Consider the configuration of pebbles in which all the pebbles are on one vertex of $W_n$, say $x$, that is not the hub. In this case, 2 pebbles are required to cover each of the three vertices adjacent to $x$, and 4 pebbles are required to cover each of the $n - 3$ vertices that are a distance of 2 away from $x$. The total number of pebbles required to cover-solve these vertices is $4n - 6$. However, we require one more pebble to place on $x$. Hence, $\gamma(W_n) \geq 4n - 5$.

To complete the proof, we will show that if there is some configuration of pebbles on $W_n$ with at least $4n - 5$ pebbles, then the configuration is cover-solvable. Suppose $C$ is a configuration of pebbles on $W_n$ and consists of at least $4n - 5$ pebbles. We now will describe a sequence of moves that will cover-solve any such configuration. First, if there is an outer vertex on $W_n$ that is empty but adjacent to another outer vertex $w$ with three or more pebbles, then make a move from $w$ to cover this vertex. Repeat this process until no empty outer vertex is adjacent to an outer vertex with three or more pebbles. Let $k$ be the number of outer vertices that are covered after this has been done.

**Case 1:** Suppose that $k = 0$. In this case, all the pebbles are on the hub vertex. To cover-solve the remaining $v - 1$ vertices, we can cover $\lfloor \frac{4v-10}{2} \rfloor = 2v - 5$ vertices using the excess pebbles already on the hub vertex. Since $v \geq 4$ and $2v-5 \geq v-1$, we can cover-solve all of the outer vertices in this manner.

**Case 2:** Suppose that $k=1$ or $k=2$. Each outer vertex covered in the process above requires at most two pebbles to cover it. Since $v \geq 4$, there are at least $4v-9-2k$ pebbles already on the hub vertex. After subtracting 1 pebble for the hub itself, there are $4v-10-2k$ pebbles that can be used such that pebbles can be placed on the remaining $v-k-1$ uncovered vertices. With these remaining pebbles on the hub, we can cover at least $\lfloor \frac{4v-10-2k}{2} \rfloor = 2v-5-k$ vertices. Since $2v-5-k \geq v-k-1$ for $v \geq 4$, there are enough pebbles to cover-solve $W_n$ in this situation.

**Case 3:** Suppose that $k \geq 3$. Again, each outer vertex in the process above requires at most two pebbles to cover it. If there are any pairs of pebbles remaining on outer vertices such that removing the pairs would not uncover that vertex, those pairs of pebbles should be moved to the hub vertex. After this process, there are at least $\lceil \frac{4v-9-2k}{2} \rceil = 2v-4-k$ pebbles on the hub vertex. Notice that this bound is based on the worst case that occurs when no pebbles are initially on the hub vertex. From the hub vertex, it takes exactly 2 pebbles to cover each of the remaining outer vertices and one pebble to cover the hub vertex. So at most $\lfloor \frac{2v-5-k}{2} \rfloor = v-3-\lfloor \frac{k}{2} \rfloor$ outer vertices can be covered. Since there are at most $v-k-1$ outer vertices left to be covered, and for $k \geq 3$, $v-k-1 \geq v-3-\lfloor \frac{k}{2} \rfloor$, there are enough pebbles to cover-solve $W_n$ in this case, and the proof is complete. □

# 4 The Cover Pebbling Number of Graphs of Diameter $d$

**Definition 4.1.** *A* binary weighting *on a graph $G$ is a function from $V(G)$ to $\{0,1\}$. If $B$ is a binary weighting on $G$, then let the order $|B|$ of $B$ be $\sum_{v \in G} B(v)$.*

**Definition 4.2.** *For a graph $G$ and binary weighting $B$, a configuration $C$ on $G$ will be called* permissible *(with respect to $B$) if for all $v \in G$, $B(v)=0 \implies C(v)=0$. A permissible configuration on a graph $G$ with a binary weighting $B$ will be called* cover-solvable *(with respect to $B$) if we can reach a configuration on which $B(v)=1 \implies C(v) \geq 1$ for all $v \in G$ by a sequence of pebbling moves.*

**Lemma 4.3.** *Let $G$ be a graph of diameter $d$, $B$ a binary weighting on $G$, and $C$ a configuration of size at least $(|B|-1)2^d+1$ on $G$ which is permissible with respect to $B$. Then $C$ is cover-solvable with respect to $B$.*

*Proof.* Assume the opposite. Then for all pairs $\{G,B\}$ of a graph $G$ together with a binary weighting on $G$ such that there exists a non-cover-solvable configuration of size at least $(|B|-1)2^d+1$ (with $d$ denoting the diameter of $G$,) choose one for which $|B|$ is minimal, and call it $\{G', B'\}$. Let $d'$ be the diameter of $G'$, let $k = (|B'| - 1)2^{d'} + 1$, and choose some configuration (call it $C'$) on $G'$ which is permissible with respect to $B'$, has size at least $k$ and is not cover-solvable.

Certainly we cannot have $|B'| = 1$, for then the only permissible configuration of size $|C'| \geq k = 1$ is the function which takes the value $|C'|$ on the lone vertex for which $B' = 1$, and is zero elsewhere. This configuration covers all vertices with non-zero weights, and so is trivially cover-solvable, creating a contradiction.

Now, suppose that $|B'| \geq 2$. If it is true that $C'(v) > 0$ whenever $B'(v) = 1$ we have a contradiction, for $C'$ is then trivially cover-solvable. Otherwise, let $v'$ be some vertex of $G'$ for which $C'(v') = 0$ and $B'(v') = 1$. At most $|B'| - 1$ vertices of $G'$ are initially occupied, and there are at least $(|B'| - 1)2^{d'} + 1$ total pebbles, so by the pigeonhole principle, there are at least $2^{d'} + 1$ pebbles on some vertex (call it $v''$). Since the diameter of $G'$ is $d'$, $d(v', v'') \leq d'$. Thus we can move $2^{d'}$ of the pebbles from $v''$ onto $v'$, through a series of pebbling moves, losing half of these pebbles for each edge we must move across, but leaving at least one pebble on $v'$ if we move all pebbles via one of the shortest paths.

Now, define a binary weighting $B^*$ on $G$ by

$$B^*(v) = \begin{cases} 0 & : \quad v = v' \\ B'(v) & : \quad v \neq v' \end{cases}$$

and define a configuration $C^*$ on $G$ by

$$C^*(v) = \begin{cases} 0 & : \quad v = v' \\ C'(v'') - 2^{d'} & : \quad v = v'' \\ C'(v) & : \quad \text{otherwise} \end{cases}$$

This is the configuration after we have moved pebbles from $v''$ onto $v'$, except that we ignore the pebbles on $v'$ and designate it as a vertex which need not be covered by pebbles. Clearly $|B^*| = |B'| - 1$ and $|C^*| = |C'| - 2^{d'}$ so from $|C'| \geq ((|B'| - 1)2^{d'} + 1)$, we see $|C^*| \geq ((|B^*| - 1)2^{d'} + 1)$. $C^*$ is permissible with respect to $B^*$, and so by our assumption of the minimality of $B'$, $C^*$ is cover-solvable with respect to $B^*$.

If we carry out the moves of the cover-solution of $C^*$ on $G$ starting with the configuration left on $G'$ after our initial movement of pebbles from $v''$ to $v'$, (certainly this is possible because this configuration is no smaller than $C^*$ on any vertex,) we will have covered every vertex of $G'$ for which $B^* = 1$. Also, we must still have $v' \geq 1$, because $C^*(v') = 0$, which does not permit any sequence of moves that decreases the number of pebbles on $v'$. Thus every vertex for which $B' = 1$ now has $C' \geq 1$, and we have cover-solved $C'$ with respect to $B'$, which contradicts the assumption that $C'$ was not cover-solvable. $\square$

**Theorem 4.4.** *Let $G$ be a graph of order $n$ and diameter $d$, and let $C$ be a configuration on $G$ of size at least $2^d(n-d+1)-1$. Then $G$ is cover-solvable (with respect to the weighting on $G$ which is equal to $1$ for each vertex.)*

*Proof.* First, we show that this bound is sharp by exhibiting the following class of graphs $G$ having $n$ vertices, diameter $d$ and $\gamma(G) = 2^d(n-d+1)-1$. Let $G_{n,d}$ be a fuse graph, which is a path on $d-1$ vertices connected to an outer vertex of a star graph containing $n-d+1$ outer vertices. By this construction, $G_{n,d}$ is of order $n$ and has diameter $d$. In [2], the cover-pebbling number of all trees is found. Thus, we know for these particular trees that $\gamma(G) = 2^d(n-d+1)-1$. Figure 1 shows an example for $n = 7$ and $d = 4$.

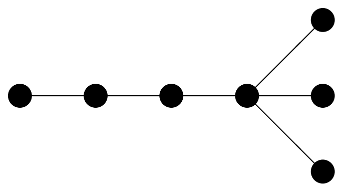

Figure 1: The graph $G_{7,4}$.

We now prove the theorem by defining an algorithm by induction which will take us to a configuration, the solvability of which we can prove using the lemma. Let $R_0 = \{v \in G : C(v) > 0\}$, let $S_0 = \{v \in G : C(v) = 0\}$, and let $T_0 = \emptyset$. Let $C_0 = C$.

For illustrative purposes, we now describe the first step of the algorithm. If $S_0 = \emptyset$, we are clearly done, for $C$ already covers $G$. Otherwise, note that since $R_0$ and $S_0$ are complementary, there exist vertices $r_0 \in R_0$ and $s_0 \in S_0$ such that $d(r_0, s_0) = 1$. If $C_0(r_0) = 1$ or $C_0(r_0) = 2$, then let $R_1 = R_0 \setminus \{r_0\}$, $S_1 = S_0$ and $T_1 = T_0 \cup \{r_0\} = \{r_0\}$. In this case, let $C_1 = C_0$.

If on the other hand $C_0(r_0) \geq 3$ then we move 2 pebbles from $r_0$ to $s_0$, and instead put $s_0$ in $T_1$ and define $C_1$ according to the following configuration. Explicitly, in this case let $R_1 = R_0$, $S_1 = S_0 \setminus \{s_0\}$, and $T_1 = T_0 \cup \{s_0\} = \{s_0\}$. Define $C_1$ on $G$ by

$$C_1(v) = \begin{cases} r_0 - 2 & : \quad v = r_0 \\ 1 & : \quad v = s_0 \\ C_0(v) & : \quad \text{otherwise} \end{cases}$$

Define the sequences $R_0, R_1, \ldots, R_{d-1}, S_0, S_1, \ldots, S_{d-1}, T_0, T_1, \ldots, T_{d-1}$, and $C_0, C_1, \ldots, C_{d-1}$, recursively in an analogous manner. Suppose for some $m < d-1$ we have $R_m$, $S_m$, $T_m$, and $C_m$, such that the following hold:

1. $|T_m| = m$.
2. $R_m$, $S_m$ and $T_m$ are disjoint and $R_m \cup S_m \cup T_m = V(G)$.
3. For all $v \in R_m \cup T_n$, $\;C_m(v) > 0$ and for all $v \in S_m$, $\;C_m(v) = 0$.
4. $C_m$ is a configuration which can be reached from $C$ by a sequence of pebbling moves.
5. $R_m$ and $S_m$ are both non-empty.
6. $\sum_{v \in R_m} C_m(v) \geq [2^d(n-d+1)-1] - [2^{m+1}-2]$.

Note that all these conditions are trivially true for $m = 0$.

From condition 1, we know that the minimum distance between $R_m$ and $S_m$ is at most $m+1$. Take points $r_m \in R_m$ and $s_m \in S_m$ for which this minimum distance is achieved (and thus $d(r_m, s_m) \leq m+1$.) If $C_m(r_m) \leq 2^{m+1}$ then let $R_{m+1} = R_m \setminus \{r_m\}$, $S_{m+1} = S_m$ and $T_{m+1} = T_m \cup \{r_m\}$. In this case, let $C_{m+1} = C_m$.

Otherwise, if $C_m(r_m) > 2^{m+1}$ then we can move $2^{m+1}$ pebbles along a minimal path from $r_m$ to $s_m$, which is of length at most $m+1$. We lose half of these pebbles for each edge we must move across, but we will be able to move $2^{(m+1)-d(r_m,s_m)} \geq 1$ onto $s_m$. Put $s_m$ in $T_{m+1}$ and define $C_{m+1}$ according to the configuration after these moves. Explicitly, in this case let $R_{m+1} = R_m$, $S_{m+1} = S_m \setminus \{s_m\}$ and $T_{m+1} = T_m \cup \{s_m\}$. Define $C_{m+1}$ on $G$ by

$$C_{m+1}(v) = \begin{cases} r_m - 2^{m+1} & : \quad v = r_m \\ 2^{(m+1)-d(r_m,s_m)} & : \quad v = s_m \\ C_m(v) & : \quad \text{otherwise} \end{cases}$$

For $m+1$, it is clear from our definitions that conditions 1, 2, 3, and 4 still hold. Condition 6 also holds, for in either of the two above cases, the total number of pebbles left on $R_{m+1}$ is at most $2^{m+1}$ less than were on $R_m$. Thus,

$$\begin{aligned} \sum_{v \in R_{m+1}} C_{m+1}(v) &\geq \sum_{v \in R_m} C_m(v) - 2^{m+1} \\ &\geq [2^d(n-d+1)-1] - [2^{m+1}-2] - 2^{m+1} \\ &= [2^d(n-d+1)-1] - [2^{m+2}-2]. \end{aligned}$$

Next we check condition 5. Since $m+1 < d$ and $n \geq d$, we know that $[2^d(n-d+1)-1] - [2^{m+1}-2] > 0$. Thus, the fact that condition 6 is true for $m+1$ necessitates that $R_m \neq \emptyset$. Also, if $S_{m+1} = \emptyset$ then $C_{m+1}(v) > 0$

for all $v \in R_m \cup S_m \cup T_m = V(G)$, and since $C_m$ is attainable from $C$ by a sequence of pebbling moves, we have cover-solved $C$ and we are done. So we may assume $S_{m+1} \neq \emptyset$ and condition 5 holds.

By this recursive definition, we now have $R_{d-1}$, $S_{d-1}$, $T_{d-1}$, and $C_{d-1}$ for which conditions 1-6 hold. Now define a binary weighting $B$ on $G$ by

$$B(v) = \begin{cases} 1 & : \quad v \in R_{d-1} \cup S_{d-1} \\ 0 & : \quad v \in T_{d-1} \end{cases}$$

Also, define $C'_{d-1}$ on $G$ by

$$C'_{d-1}(v) = \begin{cases} C_{d-1}(v) & : \quad v \in R_{d-1} \cup S_{d-1} \\ 0 & : \quad v \in T_{d-1} \end{cases}$$

Clearly $C'_{d-1}$ is permissible with respect to $B$. From condition 1, we know $|T_{d-1}| = d - 1$ so $|B| = n - d + 1$, and from condition 6 we have that $|C'_{d-1}| \geq [2^d(n-d+1)-1] - [2^{(d-1)+1} - 2] = 2^d(n-d)+1$. Thus, by the lemma, $C'_{d-1}$ is cover-solvable with respect to $B$.

By condition 4, $C_{d-1}$ is a configuration which can be reached from $C$ by a sequence of pebbling moves. If after we carry out this sequence of moves, we carry out the moves of this cover-solution of $C'_{d-1}$ on $G$ (certainly this is possible because $C'_{d-1}$ is no greater than $C_{d-1}$ on any vertex,) we will have covered every vertex of $G$ for which $B = 1$, that is every vertex in $R_{d-1} \cup S_{d-1}$. Also, every vertex $v \in T_{d-1}$ must remain covered, because for each of these vertices, $C'_{d-1}(v) = 0$, which does not permit any sequence of moves which decreases the number of pebbles on $v$. Applying, condition 2, we see for every vertex $v \in V(G) = R_{d-1} \cup S_{d-1} \cup T_{d-1}$, our final configuration after this sequence of moves is greater than zero, and so we have cover-solved $C$. □

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