

# ON THREE SETS WITH NONDECREASING DIAMETER

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ABSTRACT. Let  $[a, b]$  denote the integers between  $a$  and  $b$  inclusive and, for a finite subset  $X \subseteq \mathbb{Z}$ , let  $\text{diam}(X) = \max(X) - \min(X)$ . We write  $X <_p Y$  provided  $\max(X) < \min(Y)$ . For a positive integer  $m$ , let  $f(m, m, m; 2)$  be the least integer  $N$  such that any 2-coloring  $\Delta : [1, N] \rightarrow \{0, 1\}$  has three monochromatic  $m$ -sets  $B_1, B_2, B_3 \subseteq [1, N]$  (not necessarily of the same color) with  $B_1 <_p B_2 <_p B_3$  and  $\text{diam}(B_1) \leq \text{diam}(B_2) \leq \text{diam}(B_3)$ . Improving upon upper and lower bounds of Bialostocki, Erdős and Lefmann, we show that  $f(m, m, m; 2) = 8m - 5 + \lfloor \frac{2m-2}{3} \rfloor + \delta$  for  $m \geq 2$ , where  $\delta = 1$  if  $m \in \{2, 5\}$  and  $\delta = 0$  otherwise.

## 1. INTRODUCTION

For  $a, b \in \mathbb{R}$ , we let  $[a, b]$  denote the set of integers between  $a$  and  $b$  inclusive. For finite subsets  $X, Y \subseteq \mathbb{Z}$ , the *diameter* of  $X$ , denoted by  $\text{diam}(X)$ , is defined as  $\max(X) - \min(X)$ . Moreover, we say that  $X <_p Y$  if and only if  $\max(X) < \min(Y)$ , meaning all the elements of  $X$  come before any element from  $Y$ . For positive integers  $t, m_1, m_2, \dots, m_t, r \in \mathbb{Z}^+$ , let  $f(m_1, m_2, \dots, m_t; r)$  be the least integer  $N$  such that, for every  $r$ -coloring  $\Delta : [1, N] \rightarrow [0, r-1]$  of the integers  $[1, N]$ , there exist  $t$  subsets  $B_1, B_2, \dots, B_t \subseteq [1, N]$  with

- (a) each  $B_i$  monochromatic, i.e.,  $|\Delta(B_i)| = 1$  for  $i = 1, \dots, t$ ,
- (b)  $|B_i| = m_i$  for  $i = 1, 2, \dots, t$
- (c)  $B_1 <_p B_2 <_p \dots <_p B_t$ , and
- (d)  $\text{diam}(B_1) \leq \text{diam}(B_2) \leq \dots \leq \text{diam}(B_t)$ .

A collection of monochromatic sets  $B_i$  that satisfy (b), (c) and (d) is called a *solution* to  $p(m_1, m_2, \dots, m_t; r)$ .

The function  $f(m_1, m_2, \dots, m_t; r)$ , and the related function  $f^*(m_1, m_2, \dots, m_t; r)$  defined as  $f(m_1, m_2, \dots, m_t; r)$  but requiring the inequalities in (d) to be strict, have been studied by previous authors. Bialostocki, Erdős and Lefmann first introduced  $f(m, m, \dots, m; r)$  in [3], where they determined that  $f(m, m; 2) = 5m - 3$ , that  $f(m, m; 3) = 9m - 7$  and that

$$(1) \quad 8m - 4 \leq f(m, m, m; 2) \leq 10m - 6,$$

as well as giving asymptotic bounds for  $t = 2$ . The problem was motivated in part by zero-sum generalizations in the sense of the Erdős-Ginzburg-Ziv Theorem [6] [9, Theorem 10.1] (see [2] [3] [7] for a short discussion of zero-sum generalizations, including definitions). Subsequently, Bollobás, Erdős, and Jin obtained improved results for  $m = 2$ , showing that  $4r - \log_2 r + 1 \leq f^*(2, 2; r) \leq 4r + 1$  and  $f^*(2, 2; 2^k) = 4 \cdot 2^k + 1$ , as well as giving improved asymptotic bounds

for  $t$  and  $r$  when  $m = 2$ . The value of  $f(m, m; 4)$  was determined to be  $12m - 9$  in [8], the off diagonal cases (when not all  $m_i = m$ ) are introduced in [14], and other related Ramsey-type problems can also be found in [1] [5] [10] [11] [12] [13].

The goal of this paper is to improve the estimates from (1) to the first exact value for more than two sets. Indeed, we will show that both the upper and lower bounds of Bialostocki, Erdős and Lefmann can be improved, resulting in the value

$$f(m, m, m; 2) = 8m - 5 + \lfloor \frac{2m - 2}{3} \rfloor + \delta \quad \text{for } m \geq 2,$$

where  $\delta = 1$  if  $m \in \{2, 5\}$  and  $\delta = 0$  otherwise.

## 2. DETERMINATION OF $f(m, m, m; 2)$

Let  $\Delta : X \rightarrow C$  be a coloring of a finite set  $X$  by a set of colors  $C$ . Let  $c \in C$  and  $Y \subseteq X$ . Let  $x_1 < x_2 < \dots < x_n$  be the integers colored by  $c$  in  $Y$ . Then, for integers  $i$  and  $j$  such that  $1 \leq i \leq j \leq n$ , we use the notation  $\text{first}_i^j(c, Y)$  to denote  $\{x_i, x_{i+1}, \dots, x_j\}$ , which is the set consisting of the  $i$ -th through  $j$ -th smallest elements of  $Y$  colored by  $c$ . Likewise,  $\text{first}_i(c, Y) = x_i$  is the  $i$ -th smallest element colored by  $c$  in  $Y$ , and  $\text{first}(c, Y) = \text{first}_1(c, Y) = \min(\Delta^{-1}(c) \cap Y)$  is the first element colored by  $c$  in  $Y$ . Similarly, we define  $\text{last}_i^j(c, Y) = \{x_{n-i+1}, x_{n-i}, \dots, x_{n-j+1}\}$  to be the set consisting of the  $i$ -th through  $j$ -th largest elements of  $Y$  colored by  $c$ ,  $\text{last}_i(c, Y) = x_{n-i+1}$  to be the  $i$ -th largest element of  $Y$  colored by  $c$ , and  $\text{last}(c, Y) = \text{last}_1(c, Y) = \max(\Delta^{-1}(c) \cap Y)$  to be the last element of  $Y$  colored by  $c$ . For the sake of simplicity, a coloring  $\Delta : [1, N] \rightarrow C$  will be denoted by the string  $\Delta(1)\Delta(2)\Delta(3)\dots\Delta(N)$ , and  $x^i$  will be used to denote the string  $xx\dots x$  of length  $i$ . Hence  $\Delta : [1, 6] \rightarrow \{0, 1\}$ , where  $\Delta([1, 2]) = \{0\}$ ,  $\Delta(3) = 1$ , and  $\Delta([4, 6]) = \{0\}$ , may be represented by the string  $\Delta[1, 6] = 0^210^3$ .

The following technical lemma will help us control the possible 2-colorings of  $[1, 3m - 2]$ .

**Lemma 2.1.** *Let  $m \geq 2$ , let  $\Delta : [1, 3m - 2] \rightarrow \{0, 1\}$  be a 2-coloring and let  $B_1 \subseteq [1, 3m - 2]$  be a monochromatic  $m$ -subset with  $\text{diam}(B_1) \geq 2m - 2$  satisfying the following additional extremal constraints:*

- (a)  $\max B_1 := 3m - 2 - \beta$  is minimal, where  $\beta \in [0, m - 1]$ ;
- (b)  $\text{diam}(B_1) := 2m - 2 + \alpha$  is minimal subject to (a) holding, where  $\alpha \in [0, m - 1 - \beta]$ .

Suppose  $B_1$  exists and  $\Delta(B_1) = \{1\}$ . Then, letting  $R = [1, 3m - 2 - \beta]$ , one of the following holds.

- (i)
  - $\beta \leq m - 2$  and  $|\Delta^{-1}(0) \cap R| \geq m$
  - $\Delta R = 1^{m-1-\beta-\nu}H_00H_11^{1+\nu}$ , where  $\mu, \nu \geq 0$  are integers
  - $\beta = m - 1 - \alpha$  or  $\nu = \mu = 0$
  - $H_1$  is a string of length  $m - 2 - \beta + \mu$  with exactly  $\mu$  1's and exactly  $m - 2 - \beta$  0's
  - $H_0$  is a string of length  $m - 1 + \beta - \mu$  containing exactly  $m - 1 - \alpha - \mu$  1's and exactly  $\alpha + \beta$  0's

- (ii) • either  $\beta < m - 1 - \alpha$  or  $\beta = m - 1$   
 •  $\Delta R = 0^{m-\alpha-\beta-1}1H_21^{m-\beta}$   
 •  $H_2$  is a string of length  $m - 2 + \beta + \alpha$   
 • if  $\beta \leq m - 2$ , then  $|\Delta^{-1}(0) \cap R| \geq m$   
 • if  $\alpha > 0$ , then  $\beta \geq 1$  and  $H_2$  contains exactly  $\beta - 1$  1's
- (iii) •  $\beta \geq \alpha$   
 •  $|\Delta^{-1}(0) \cap R| < m$   
 •  $\text{first}_m(1, R) \leq 3m - 3 - \beta - \alpha$   
 •  $|\Delta^{-1}(0) \cap [\text{first}_1(1, R), \text{first}_m(1, R)]| \leq \beta$

*Proof.* Note (a) and (b) imply

$$(2) \quad \min B_1 = \max B_1 - \text{diam } B_1 = 3m - 2 - \beta - (2m - 2 + \alpha) = m - \alpha - \beta.$$

Let

$$\eta = 3m - 3 - \beta - \text{last}_2(1, R) \geq 0$$

be the number of integers colored by 0 between  $\text{last}_1(1, R)$  and  $\text{last}_2(1, R)$ . Let

$$\nu = 3m - 3 - \beta - \text{last}(0, R) \geq 0$$

be the number of integers strictly between  $3m - 2 - \beta$  and  $\text{last}(0, R)$  that are colored by 1. We continue with three claims.

**Claim A.** If  $|\Delta^{-1}(1) \cap R| > m$ , then  $\text{last}_2(1, R) \leq 3m - 3 - \beta - \alpha$  and  $\eta \geq \alpha$ .

If we have  $\text{last}_2(1, R) \geq 2m - 2 + \text{first}(1, R)$ , then  $B'_1 = \text{first}_1^{m-1}(1, R) \cup \{\text{last}_2(1, R)\}$  will be a monochromatic  $m$ -subset, in view of the hypothesis  $|\Delta^{-1}(1) \cap R| > m$ , with  $\max B'_1 < \max B_1$  and  $\text{diam } B'_1 = \text{last}_2(1, R) - \text{first}(1, R) \geq 2m - 2$ , contradicting the maximality condition (a) for  $B_1$ . Therefore we may instead assume  $\text{last}_2(1, R) \leq 2m - 3 + \text{first}(1, R) \leq 3m - 3 - \beta - \alpha$ , where the final inequality follows from  $\text{first}(1, R) \leq \min B_1$  and (2), and now  $\eta \geq \alpha$  follows from the definition of  $\eta$ , completing the claim.

**Claim B.** If  $|\Delta^{-1}(0) \cap R| < m$ , then either  $\beta = m - 1$  and (ii) holds or else (iii) holds.

If  $\beta = m - 1$ , then  $\alpha = 0$  in view of  $\alpha \in [0, m - 1 - \beta]$ . Moreover,  $\min B_1 = 1$  by (2), and now (ii) is easily seen to hold. Therefore we may assume  $\beta \leq m - 2$ , in which case

$$(3) \quad |\Delta^{-1}(1) \cap R| = 3m - 2 - \beta - |\Delta^{-1}(0) \cap R| \geq 2m - |\Delta^{-1}(0) \cap R| \geq m + 1,$$

where we have utilized the claim hypothesis for the final inequality. Thus Claim A implies

$$(4) \quad \text{last}_2(1, R) \leq 3m - 3 - \beta - \alpha \quad \text{and} \quad \eta \geq \alpha.$$

Moreover, since  $\text{first}_m(1, R) \leq \text{last}_2(1, R)$  by (3), we see that (4) also implies

$$\text{first}_m(1, R) \leq 3m - 3 - \beta - \alpha.$$

By the hypothesis of the claim, we have

$$(5) \quad |\Delta^{-1}(1) \cap [m - \alpha - \beta, 3m - 2 - \beta]| \geq 2m - 1 + \alpha - |\Delta^{-1}(0) \cap R| \geq m + \alpha.$$

We must have

$$(6) \quad \Delta([1, m - 1 - \beta - \eta]) \subseteq \{0\},$$

for if  $\text{first}(1, R) \leq m - 1 - \beta - \eta$ , then  $B'_1 = \text{first}_1^{m-1}(1, R) \cup \{\text{last}_2(1, R)\}$  will be a monochromatic  $m$ -subset in view of (3) with  $\max B'_1 < \max B_1$  and  $\text{diam } B'_1 \geq \text{last}_2(1, R) - (m - 1 - \beta - \eta) = 2m - 2$  (in view of the definition of  $\eta$ ), contradicting the extremal condition (a) for  $B_1$ .

From (6), we see there are at least  $m - 1 - \beta - \eta$  integers colored by 0 less than  $\text{first}(1, R)$  (with this estimate being rather trivial when  $[1, m - 1 - \beta - \eta] = \emptyset$ ). By the definition of  $\eta$ , we have at least  $\eta$  integers colored by 0 all greater than  $\text{last}_2(1, R)$ . In particular, since (3) implies  $\text{first}_m(1, R) \leq \text{last}_2(1, R)$ , we find that there are at least  $\eta$  integers colored by 0 greater than  $\text{first}_m(1, R)$ . Since  $|\Delta^{-1}(0) \cap R| < m$  holds by the claim hypothesis, this leaves at most  $m - 1 - \eta - (m - 1 - \beta - \eta) = \beta$  integers that can be colored by 0 between  $\text{first}(1, R)$  and  $\text{first}_m(1, R)$ , i.e.,

$$|\Delta^{-1}(0) \cap [\text{first}_1(1, R), \text{first}_m(1, R)]| \leq \beta.$$

It now remains to show

$$\beta \geq \alpha$$

and then (iii) will follow, completing the claim. If  $\alpha = 0$ , then this holds trivially, so we now assume  $\alpha \geq 1$ , in which case (5) implies there are at least  $m + 1$  integers colored by 1 in the interval  $[m - \alpha - \beta, 3m - 2 - \beta]$ . Recall from (2) that  $\min B_1 = m - \alpha - \beta$ . Then we must have

$$(7) \quad \Delta([m - \alpha - \beta + 1, m - \beta]) = \{0\},$$

for otherwise

$$B'_1 = \text{first}_2^m(1, [m - \alpha - \beta, 3m - 2 - \beta]) \cup \{3m - 2 - \beta\}$$

will be a monochromatic  $m$ -subset with  $\text{diam } B_1 > \text{diam } B'_1 \geq 3m - 2 - \beta - m + \beta = 2m - 2$ , contradicting the extremal condition (b) for  $B_1$ . But now we have at least  $m - 1 - \beta - \eta$  integers colored by 0 in  $[1, m - 1 - \beta - \eta]$  by (6), at least  $\alpha$  integers colored by 0 in  $[m - \alpha - \beta + 1, m - \beta]$  by (7), and  $\eta$  integers colored by 0 in  $[3m - 2 - \beta - \eta, 3m - 3 - \beta]$  by the definition of  $\eta$ . Since

$$m - 1 - \beta - \eta < m - \alpha - \beta + 1 \leq m - \beta < 3m - 2 - \beta - \eta,$$

where the first inequality follows in view (4), the second from the assumption  $\alpha \geq 1$  (noted at the start of the paragraph), and the third from the trivial inequality  $\eta \leq |\Delta^{-1}(0) \cap R|$  combined with the claim's hypothesis  $|\Delta^{-1}(0) \cap R| \leq m - 1$ , it follows that these three intervals are all disjoint. Thus

$$|\Delta^{-1}(0) \cap R| \geq (m - 1 - \beta - \eta) + \alpha + \eta = m - 1 - \beta + \alpha.$$

But combining the above inequality with the claim hypothesis  $|\Delta^{-1}(0) \cap R| \leq m - 1$  now yields  $\beta \geq \alpha$ , completing the claim as remarked previously.

**Claim C.** If  $\eta > 0$  and  $|\Delta^{-1}(0) \cap R| \geq m$ , then

$$(8) \quad \Delta([1, m - 1 - \beta]) \subseteq \{1\}.$$

Moreover, if we also have  $\beta \leq m - 2$  and  $|\Delta^{-1}(1) \cap R| > m$ , then  $\eta \geq m - 1 - \beta$ .

From the first hypothesis  $\eta > 0$  and the definition of  $\eta$ , we have  $\Delta(3m - 3 - \beta) = 0$ . Thus from the second hypothesis  $|\Delta^{-1}(0) \cap R| \geq m$ , we conclude that (8) holds, for otherwise  $B'_1 = \text{first}_1^{m-1}(0, R) \cup \{3m - 3 - \beta\}$  will be a monochromatic  $m$ -subset with  $\max B'_1 < \max B_1$  and  $\text{diam } B'_1 \geq 3m - 3 - \beta - (m - 1 - \beta) = 2m - 2$ , contradicting the extremal condition (a) for  $B_1$ . This completes the first part of the claim, and we now assume the hypotheses of the second part. In view of (8) and the third hypothesis  $\beta \leq m - 2$ , it follows that  $\Delta(1) = 1$ , and now we must have

$$\Delta([2m - 1, 3m - 3 - \beta]) = \{0\},$$

for otherwise  $B'_1 = \text{first}_1^{m-1}(1, R) \cup \{\text{last}_2(1, R)\}$  will be a monochromatic  $m$ -subset, in view of the fourth hypothesis  $|\Delta^{-1}(1) \cap R| > m$ , with  $\max B'_1 < \max B_1$  and  $\text{diam } B'_1 \geq (2m - 1) - 1 = 2m - 2$ , contradicting the extremal condition (a) for  $B_1$ . The claim now follows in view of the definition of  $\eta$ .

In view of Claim B, we may assume

$$(9) \quad |\Delta^{-1}(0) \cap R| \geq m,$$

else the proof is complete. We divide the remainder of the proof into two cases.

**Case 1:**  $\beta = m - 1 - \alpha$ . Note this is equivalent to  $\min B_1 = 1$  in view of (2).

If  $\alpha = 0$ , then  $\beta = m - 1$  and (ii) follows. So assume

$$\alpha \geq 1 \quad \text{and} \quad \beta \leq m - 2,$$

where the latter inequality follows from the former in view of the case hypothesis. We will show (i) holds.

Suppose  $|\Delta^{-1}(1) \cap R| = m$ . Then  $|\Delta^{-1}(0) \cap R| = 2m - 2 - \beta$ . In view of (9), we see that, if  $|\Delta^{-1}(1) \cap [\text{first}(0, R), \text{last}(0, R)]| > \beta$ , then  $B'_1 = \text{first}_1^{m-1}(0, R) \cup \{\text{last}_1(0, R)\}$  will be a monochromatic  $m$ -subset with  $\text{diam}(B'_1) \geq |\Delta^{-1}(0) \cap R| - 1 + \beta + 1 = 2m - 2$  and  $\max B'_1 < \max B_1$ , contradicting the maximality condition (a) for  $B_1$ . Therefore we must instead have  $|\Delta^{-1}(1) \cap [\text{first}(0, R), \text{last}(0, R)]| \leq \beta$ . From the definition of  $\nu$ , there are precisely  $\nu + 1$  integers colored by 1 to the right of  $\text{last}(0, R) = 3m - 3 - \beta - \nu$  in  $R$ . Combined with the previous sentence, this means there are at most  $\beta + \nu + 1$  elements of  $R$  colored by 1 greater than  $\text{first}(0, R)$ . But now, since  $|\Delta^{-1}(1) \cap R| = m$ , there must be at least  $m - 1 - \beta - \nu$  integers colored by 1 to the left of  $\text{first}(0, R)$ . Consequently, if  $m - 1 - \beta - \nu \geq 1$ , then (i) follows by letting  $H_1$  be the string given by  $\Delta[\text{last}_{m-1-\beta}(0, R) + 1, 3m - 3 - \beta - \nu]$ , letting  $\mu$  be the number of 1's in

$H_1$ , and noting that the case hypothesis gives  $\beta = m - 1 - \alpha$  (so that  $H_0$  containing exactly  $m - 1 - \alpha - \mu$  1's is equivalent to  $|\Delta^{-1}(1) \cap R| = m$ ). On the other hand, if  $m - 1 - \beta - \nu \leq 0$ , then let  $\nu' = m - 2 - \beta < \nu$ . Since  $\beta \leq m - 2$ , we also have  $\nu' \geq 0$ , while  $m - 1 - \beta - \nu' = 1$ , which is colored by 1 in view of  $\min B_1 = 1$ . Thus (i) follows using  $\nu'$  in place of  $\nu$  by letting  $H_1$  be the string given by  $\Delta[\text{last}_{m-1-\beta}(0, R) + 1, 3m - 3 - \beta - \nu']$ , and by letting  $\mu$  be the number of 1's in  $H_1$ . So we may now assume

$$(10) \quad |\Delta^{-1}(1) \cap R| > m.$$

In particular, (10) and (9) together force  $3m - 2 - \beta = |R| \geq m + 1 + m$ , implying  $\beta \leq m - 3$ .

In view of (10) and Claim A, it follows that  $\eta \geq \alpha > 0$ , which, together with (9), allows us to apply Claim C to conclude  $\Delta([1, m - 1 - \beta]) = \{1\}$ . Thus, in view of  $\beta \leq m - 3$ , we find that  $\Delta(1) = \Delta(2) = 1$ , i.e.,  $\text{first}_2(1, R) = 2$ . As a result,  $B'_1 = \text{first}_2^m(1, R) \cup \{3m - 2 - \beta\}$  is a monochromatic  $m$ -subset (in view of (10)) with

$$\text{diam } B'_1 = 3m - 4 - \beta = 2m - 3 + \alpha \geq 2m - 2,$$

where the second equality follows by the case hypothesis and the inequality from the assumption  $\alpha \geq 1$ . Since  $\max B'_1 = \max B_1$  and  $2m - 2 \leq \text{diam } B'_1 < \text{diam } B_1 = 2m - 2 + \alpha$ , this contradicts the extremal condition (b) for  $B_1$ , completing the case.

**Case 2:**  $\beta < m - 1 - \alpha$ . Note this is equivalent to  $\min B_1 > 1$  by (2) and implies

$$\beta \leq m - 2$$

in view of  $\alpha \geq 0$ . We divide this case into two subcases.

**Subcase 2.1:**  $\eta = 0$ , so that  $\text{last}_2(1, R) = 3m - 3 - \beta$ .

We will show (ii) holds. From the subcase hypothesis and Claim A, it follows that

$$\alpha = 0 \quad \text{or} \quad |\Delta^{-1}(1) \cap R| = m.$$

From (2), we know there are  $m$  integers in  $R$  colored by 1 all at least  $m - \alpha - \beta$ , namely, the  $m$  integers from  $B_1$ . Thus, if  $\text{first}(1, R) < m - \alpha - \beta$ , then the set  $B'_1 = \text{first}_1^{m-1}(1, R) \cup \{\text{last}_2(1, R)\}$  will be a monochromatic  $m$ -subset with  $\text{diam } B'_1 \geq 3m - 3 - \beta - (m - \alpha - \beta - 1) = 2m - 2 + \alpha \geq 2m - 2$  and  $\max B'_1 < \max B_1$ , contradicting the extremal condition (a) for  $B_1$ . Therefore we may instead assume  $\text{first}(1, R) = \min B_1 = m - \alpha - \beta$  (in view of (2)), so that

$$(11) \quad \Delta([1, m - \alpha - \beta - 1]) = \{0\} \quad \text{and} \quad \Delta(m - \alpha - \beta) = 1.$$

Now  $[1, m - \alpha - \beta - 1]$  is a nonempty interval (in view of the hypothesis of Case 2) entirely colored by 0. Consequently, in view of (9), it follows that the extremal condition (a) for  $B_1$  will be contradicted by  $\text{first}_1^{m-1}(0, R) \cup \{\text{last}(0, R)\}$  unless  $\Delta([2m - 1, 3m - 2 - \beta]) = \{1\}$ . However, in this latter case, (ii) follows (note  $H_2$  containing exactly  $\beta - 1$  1's is equivalent to  $|\Delta^{-1}(1) \cap R| = m$ , and that this in turn forces  $\beta - 1 \geq 0$ ), completing the subcase.

**Subcase 2.2:**  $\eta > 0$ 

We will show (i) holds with  $\nu = \mu = 0$  in this case. In view of the subcase hypothesis  $\eta > 0$  and (9), we may apply Claim C to conclude that

$$(12) \quad \Delta([1, m-1-\beta]) = \{1\}.$$

In particular, in view of  $\beta \leq m-2$ , we see that  $\Delta(1) = 1$ . Thus, we must have

$$(13) \quad |\Delta^{-1}(1) \cap R| > m,$$

for otherwise  $\min B_1 = 1$  by (2), contrary to the hypothesis of Case 2. But now we can further apply Claim C to conclude

$$\eta \geq m-1-\beta \quad \text{and} \quad \Delta([2m-1, 3m-3-\beta]) = \{0\},$$

where the second statement above is simply a restatement of the first in view of the definition of  $\eta$ .

If  $\alpha = 0$ , then  $\min B_1 = m-\beta$  by (2), in which case (12) shows that all integers colored by 0 in  $R$  lie between  $\min B_1$  and  $\max B_1 = 3m-2-\beta$ . Thus, in view of (9), it follows that  $\text{diam } B_1 \geq |B_1| - 1 \geq 2m-1$ , contradicting that  $\text{diam } B_1 = 2m-2+\alpha = 2m-2$  by (b). Therefore we instead conclude that  $\alpha \geq 1$ .

If  $\alpha = 1$ , then  $\min B_1 = m-\alpha-\beta = m-1-\beta$  by (2). Thus (12) shows that all integers colored by 0 in  $R$  lie between  $\min B_1$  and  $\max B_1 = 3m-2-\beta$ . Hence  $2m-1 = 2m-2+\alpha = \text{diam } B_1 \geq m-1 + |\Delta^{-1}(0) \cap R|$ , which combined with (9) forces there to be exactly  $|\Delta^{-1}(0) \cap R| = m$  integers colored by 0 in  $[\min B_1 + 1, \max B_1 - 1] = [m-\beta, 3m-3-\beta]$ , and thus exactly  $m-2 = m-1-\alpha$  integers colored by 1 in  $[m-\beta, 3m-3-\beta]$ . As a result, (i) follows with  $\mu = \nu = 0$ . Therefore we may now assume  $\alpha \geq 2$ .

Since  $\alpha \geq 2$ , (12) implies that  $\Delta(m-\alpha-\beta+1) = 1$ . As a result, recalling from (2) that  $\min B_1 = m-\alpha-\beta$ , we must have

$$|\Delta^{-1}(1) \cap [m-\alpha-\beta, 3m-2-\beta]| = m,$$

for otherwise  $B'_1 = \text{first}_2^m(1, [m-\alpha-\beta, 3m-2-\beta]) \cup \{3m-2-\beta\}$  will be a monochromatic  $m$ -subset with  $\max B'_1 = \max B_1$  and  $\text{diam } B'_1 = \text{diam } B_1 - 1 = 2m-3+\alpha \geq 2m-2$ , contradicting the extremal condition (b) for  $B_1$ . But now (i) follows with  $\nu = \mu = 0$ , completing the proof.  $\square$

The next lemma translates the structural information from Lemma 2.1 into the existence of sets with small diameter.

**Lemma 2.2.** *Let  $m \geq 2$  and let  $\Delta : [1, 3m-2] \rightarrow \{0, 1\}$  be a 2-coloring.*

- (i) *If there does not exist a monochromatic  $m$ -subset  $B$  such that  $\text{diam}(B) \geq 2m-2$ , then there exist monochromatic  $m$ -subsets  $D_1, D_2 \subseteq [1, 3m-2]$  such that*

$$D_1 <_p D_2 \quad \text{and} \quad \text{diam}(D_1) = \text{diam}(D_2) = m-1.$$

- (ii) *Otherwise, if  $\beta, \alpha, \nu, \mu$ , and  $B_1$  are as defined in Lemma 2.1, then the following hold.*

- (a) *There exist monochromatic  $m$ -subsets  $A_1, A_2 \subseteq [1, 3m - 2 - \alpha - \beta]$  with*
- $$\text{diam}(A_1) \leq 2m - 2 - \alpha \quad \text{and} \quad \text{diam}(A_2) \leq m + \lfloor \frac{m-1+\beta}{2} \rfloor - 1.$$
- (b) *If either Lemma 2.1(iii) holds, or  $\alpha \geq 1$  and Lemma 2.1(ii) holds, then*

$$\text{diam } A_1 \leq m - 1 + \beta.$$

- (c) *If Lemma 2.1(i) holds, then*

$$\text{diam } A_1 \leq 2m - 2 - \alpha - \mu$$

*and there exists a monochromatic  $m$ -subset  $A_3 \subseteq [1, m + \alpha + \beta] \subseteq [1, 3m - 2 - \alpha - \beta]$  with*

$$\text{diam } A_3 \leq m + \alpha + \beta - 1.$$

*Proof.* First suppose that there does not exist a monochromatic  $m$ -subset  $B$  with  $\text{diam}(B) \geq 2m - 2$ . We may assume each color is used at least  $m$  times, for otherwise w.l.o.g.  $D = \text{first}_1^{m-1}(1, [1, 3m-2]) \cup \{\text{last}(1, [1, 3m-2])\}$  is a monochromatic  $m$ -set with  $\text{diam}(D) \geq |D| - 1 \geq 2m - 2$ , contrary to hypothesis. We may w.l.o.g. assume  $\Delta(1) = 1$ . Since there does not exist a monochromatic  $m$ -subset  $B$  with  $\text{diam}(B) \geq 2m - 2$ , and since each color is used at least  $m$  times, it follows that  $\text{last}(1, [1, 3m-2]) \leq 2m-2$ ,  $\Delta(3m-2) = 0$  and  $\text{first}(0, [1, 3m-2]) \geq m+1$ . Hence  $D_1 = [1, m]$  and  $D_2 = [2m-1, 3m-2]$  satisfy (i).

So we may now assume there exists a monochromatic  $m$ -subset  $B$  with  $\text{diam}(B) \geq 2m - 2$ . Let  $\beta, \alpha, \nu, \mu$ , and  $B_1$  be as defined in Lemma 2.1. Let  $R = [1, 3m - 2 - \beta]$  and assume w.l.o.g.  $\Delta(B_1) = \{1\}$ . Notice that  $\beta \leq m - 1$  implies  $\beta \leq \lfloor \frac{m-1+\beta}{2} \rfloor$ , so that

$$m - 1 + \beta \leq m + \lfloor \frac{m-1+\beta}{2} \rfloor - 1.$$

Applying Lemma 2.1 to  $[1, 3m - 2]$  yields three cases.

**Case 1:** Lemma 2.1(i) holds.

Suppose  $\beta = m - 1 - \alpha$ . Then  $H_0$  contains exactly  $m - 1 - \alpha - \mu$  1's and exactly  $\alpha + \beta = m - 1$  0's. Thus the string  $H_0 0$  contains  $m$  0's and exactly  $m - 1 - \alpha - \mu = \beta - \mu$  1's, in which case  $A_1 = A_2 = A_3 = \text{first}_1^m(0, R) \subseteq [1, 2m - 1 - \mu - \nu] \subseteq [1, m + \alpha + \beta] = [1, 3m - 2 - \alpha - \beta]$  is a monochromatic  $m$ -subset with  $\text{diam } A_1 \leq 2m - 2 - \alpha - \mu = m - 1 + \beta - \mu$ , as desired. So we may now assume  $\beta < m - 1 - \alpha$ , in which case

$$(14) \quad \mu = \nu = 0 \quad \text{and} \quad \Delta[1, 3m - 2 - \alpha - \beta] = 1^{m-1-\beta} H_0 0^{m-\alpha-\beta}.$$

Since  $H_0$  contains exactly  $m - 1 - \alpha$  1's and exactly  $\alpha + \beta$  0's, it follows from (14) that  $A_1 = \text{first}_1^m(0, [1, 3m - 2 - \alpha - \beta])$  is a monochromatic  $m$ -subset such that  $\text{diam } A_1 \leq m - 1 + (m - 1 - \alpha) = 2m - 2 - \alpha$ . Moreover, since  $m - 1 - \beta + m - 1 - \alpha = 2m - 2 - \alpha - \beta \geq m$  in view of  $\beta < m - \alpha - 1$ , it follows that there are at least  $m$  1's in the string  $1^{m-1-\beta} H_0$ . Consequently, since there are at most  $\alpha + \beta$  0's in  $H_0$ , it follows that  $A_3 = \text{first}_1^m(1, [1, m + \alpha + \beta])$  is a



monochromatic  $m$ -subset with  $\text{diam } A_3 \leq m + \alpha + \beta - 1$  and  $A_3 \subseteq [1, m + \alpha + \beta]$ . Since  $\frac{1}{2}(\text{diam } A_1 + \text{diam } A_3) \leq \frac{1}{2}(3m - 3 + \beta) = m + \frac{m-1+\beta}{2} - 1$ , we can take  $A_2$  to be the set from  $A_1$  or  $A_3$  having smaller diameter, and then  $\text{diam } A_2 \leq m + \lfloor \frac{m-1+\beta}{2} \rfloor - 1$ , completing the case.

**Case 2:** Lemma 2.1(ii) holds.

If  $\alpha > 0$ , then  $\beta \leq m - 1 - \alpha \leq m - 2$ , and letting  $A_1 = A_2 = \text{first}_1^m(0, R) \subseteq [1, m + \beta] \subseteq [1, 3m - 2 - \alpha - \beta]$ , we find that  $\text{diam } A_1 \leq m - 1 + \beta \leq 2m - 2 - \alpha$ , as desired. It remain to consider when  $\alpha = 0$ .

Taking  $A_1 = B_1 \subseteq [1, 3m - 2 - \beta] = [1, 3m - 2 - \alpha - \beta]$  gives a monochromatic  $m$ -subset with  $\text{diam } A_1 = \text{diam } B_1 = 2m - 2 + \alpha = 2m - 2 - \alpha$ . If  $\beta = m - 1$ , then  $m + \lfloor \frac{m-1+\beta}{2} \rfloor - 1 = 2m - 2$ , and we may take  $A_2 = A_1$ . Therefore assume  $\beta < m - 1$ . Assume by contradiction that there is no monochromatic  $m$ -subset  $A_2 \subseteq [1, 3m - 2 - \alpha - \beta] = R$  with  $\text{diam } A_2 \leq m + \lfloor \frac{m-1+\beta}{2} \rfloor - 1$ . Since  $\Delta([2m - 1, 3m - 2 - \beta]) = \{1\}$ , this implies

$$(15) \quad |\Delta^{-1}(1) \cap [2m - \lfloor \frac{m-1+\beta}{2} \rfloor - 1 - \beta, 2m - 2]| \leq \beta - 1,$$

else  $A_2 = \text{last}_1^m(1, R)$  will be such a set. On the other hand, since  $\Delta([1, m - 1 - \beta]) = \{0\}$  and  $|\Delta^{-1}(0) \cap R| \geq m$ , it likewise follows that

$$(16) \quad |\Delta^{-1}(0) \cap [m - \beta, m + \lfloor \frac{m-1+\beta}{2} \rfloor]| \leq \beta,$$

else  $A_2 = \text{first}_1^m(0, R)$  will be such a set. Observe that

$$\begin{aligned} 2m - \lfloor \frac{m-1+\beta}{2} \rfloor - 1 - \beta &\geq 2m - \lfloor \frac{m-1+m-2}{2} \rfloor - 1 - \beta = m - \beta + 1 \quad \text{and} \\ m + \lfloor \frac{m-1+\beta}{2} \rfloor &\leq m + \lfloor \frac{m-1+m-2}{2} \rfloor = 2m - 2, \end{aligned}$$

both in view of  $\beta \leq m - 2$ . Thus, letting  $R' = [2m - \lfloor \frac{m-1+\beta}{2} \rfloor - 1 - \beta, m + \lfloor \frac{m-1+\beta}{2} \rfloor]$ , we see that (15) and (16) yield the contradiction

$$2\beta \leq 2\lfloor \frac{m-1+\beta}{2} \rfloor - m + 2 + \beta = |R'| \leq 2\beta - 1,$$

completing the case.

**Case 3:** Lemma 2.1(iii) holds.

Letting  $\text{first}_1^m(1, [1, 3m - 2 - \beta - \alpha]) = A_1 = A_2$ , it follows that  $\text{diam}(A_1) \leq m - 1 + \beta \leq 2m - 2 - \alpha$ , completing the proof.  $\square$

We are now ready to proceed with our main result.

**Theorem 2.1.** *Let  $m \geq 2$  be an integer. Then*

$$f(m, m, m; 2) = 8m - 5 + \lfloor \frac{2m-2}{3} \rfloor + \delta,$$

where  $\delta = 1$  if  $m \in \{2, 5\}$  and  $\delta = 0$  otherwise.

*Proof.* Observe that  $f(m, m, m; 2) \geq 8m - 5 + \lfloor \frac{2m-2}{3} \rfloor$  follows for  $m \geq 2$  by considering the coloring of  $[1, 8m - 6 + \lfloor \frac{2m-2}{3} \rfloor]$  given by the string

$$01^{m-1}0^{m-1}1^{m-1}0^{\lfloor \frac{2m-2}{3} \rfloor}1^{m-\lfloor \frac{2m-2}{3} \rfloor-1}0^{m-1}1^{2m-1+\lfloor \frac{2m-2}{3} \rfloor}0^{m-1}.$$

Likewise,  $f(2, 2, 2; 2) \geq 12$  follows by considering the coloring of  $[1, 11]$  given by the string

$$10101101110,$$

and  $f(5, 5, 5; 2) \geq 38$  follows by considering the coloring of  $[1, 37]$  given by the string

$$01^40^41^40^81^40^21^70^3.$$

We proceed to show that  $f(m, m, m; 2) \leq 8m - 5 + \lfloor \frac{2m-2}{3} \rfloor + \delta$ , where  $\delta = 1$  for  $m \in \{2, 5\}$  and  $\delta = 0$  otherwise.

Suppose by contradiction that  $\Delta[-2m + 2, 6m - 4 + \lfloor \frac{2m-2}{3} \rfloor + \delta] \rightarrow \{0, 1\}$  is a 2-coloring that avoids a monochromatic solution to  $p(m, m, m; 2)$  (the problem is translation invariant, so we choose our interval to begin with  $-2m + 2$  for notational convenience in the proof). From the pigeonhole principle, it follows that there is a monochromatic  $m$ -set

$$A_0 \subseteq [-2m + 2, 0] \quad \text{with} \quad \text{diam}(A_0) \leq 2m - 2.$$

The strategy is to show that any 2-coloring of  $[1, 6m - 4 + \lfloor \frac{2m-2}{3} \rfloor + \delta]$  must either contain a monochromatic solution to  $p(m, m, m; 2)$  or else a monochromatic solution  $B <_p C$  to  $p(m, m; 2)$  with  $\text{diam} B \geq 2m - 2$ . In the latter case,  $A_0$ ,  $B$  and  $C$  will give the desired monochromatic solution to  $p(m, m, m; 2)$ .

Apply Lemma 2.2 to  $[1, 3m - 2]$ . If (i) holds, then, since the pigeonhole principle guarantees there is a monochromatic  $m$ -set  $D_3 \subseteq [3m - 1, 5m - 3]$ , we find that  $D_1$ ,  $D_2$  and  $D_3$  form a monochromatic solution to  $p(m, m, m; 2)$ . So we may assume (ii) of Lemma 2.2 holds and we will apply it and Lemma 2.1 to  $[1, 3m - 2]$ . Let  $\alpha, \beta, \nu, \mu, A_1, A_2, A_3$  and  $B_1$  be as in Lemmas 2.2 and 2.1. Let  $\Delta(B_1) = \{c_1\}$  with  $\{c_1, c_0\} = \{1, 0\}$ . Thus, when reading the conclusion of Lemma 2.1, we must use  $c_1$  in place of 1 and  $c_0$  in place of 0. Recall that we have the trivial inequality

$$0 \leq \alpha + \beta \leq m - 1$$

in view of the definition of  $\alpha$ . Let

$$R_1 = [1, 3m - 2 - \beta] \quad \text{and} \quad R_2 = [3m - 1 - \beta, 6m - 4 + \lfloor \frac{2m-2}{3} \rfloor + \delta].$$

**Step 1:**  $|\Delta^{-1}(0) \cap R_2| \geq m$  and  $|\Delta^{-1}(1) \cap R_2| \geq m$ .

Suppose w.l.o.g. that  $|\Delta^{-1}(0) \cap R_2| < m$ . Then

$$|\Delta^{-1}(1) \cap R_2| \geq 2m - 1 + \beta + \lfloor \frac{2m-2}{3} \rfloor + \delta.$$

Let

$$\gamma = |\Delta^{-1}(0) \cap [\text{first}(1, R_2), \text{last}(1, R_2)]|.$$

If  $\alpha \leq \lfloor \frac{2m-2}{3} \rfloor + \gamma + \beta + \delta$ , then  $A_0$ ,  $B_1$ , and  $C = \text{first}_1^{m-1}(1, R_2) \cup \text{last}(1, R_2)$  form a monochromatic solution to  $p(m, m, m; 2)$ . So we may assume

$$(17) \quad \alpha \geq \lfloor \frac{2m-2}{3} \rfloor + \gamma + \beta + \delta + 1 \geq \frac{2m-1}{3} + \delta + \gamma + \beta.$$

Let  $y$  be the least integer such that  $\Delta(y) = 1$ ,  $y \geq \text{first}(1, R_2) + 2m - 2 - \alpha$  and  $y \geq \text{first}_m(1, R_2)$ , and let

$$C_1 = \text{first}_1^{m-1}(1, R_2) \cup \{y\}.$$

Note  $y$  exists in view of  $|\Delta^{-1}(1) \cap R_2| \geq 2m - 1 - \alpha \geq m$ . Observe that

$$(18) \quad |\Delta^{-1}(1) \cap [\text{first}(1, R_2), y]| \leq \max\{2m - 1 - \alpha, m\} = 2m - 1 - \alpha,$$

since otherwise the minimality of  $y$  is contradicted by  $y' = \text{first}_{2m-1-\alpha}(1, R_2)$ . Then

$$(19) \quad 2m - 2 - \alpha \leq \text{diam } C_1 \leq 2m - 2 - \alpha + \gamma \leq \frac{4m-5}{3} - \delta - \beta,$$

where the latter inequality follows from (17). By (18), there are at least

$$2m - 1 + \beta + \lfloor \frac{2m-2}{3} \rfloor + \delta - (2m - 1 - \alpha) = \lfloor \frac{2m-2}{3} \rfloor + \alpha + \beta + \delta$$

integers colored by 1 in

$$R_3 = [y + 1, 6m - 4 + \lfloor \frac{2m-2}{3} \rfloor + \delta].$$

In view of (17), we have  $\lfloor \frac{2m-2}{3} \rfloor + \alpha + \beta + \delta \geq m$ . Thus, letting  $C_2 = \text{first}_1^{m-1}(1, R_3) \cup \{\text{last}(1, R_3)\}$ , it follows from (17) that

$$\text{diam } C_2 \geq |\Delta^{-1}(1) \cap R_3| - 1 \geq \frac{2m-4}{3} + \alpha + \beta + \delta - 1 \geq \frac{4m-5}{3} + 2\delta + 2\beta + \gamma - 1 \geq \frac{4m-5}{3} - 1.$$

Comparing the above bound with that of (19), we see that if any estimate used in obtaining these bounds can be improved by 1, then  $A_1$ ,  $C_1$  and  $C_2$  will be a monochromatic solution to  $p(m, m, m; 2)$ . Assuming this is not the case, we instead find that  $\delta = 0$ ,  $\beta = 0$ ,  $\gamma = 0$ , and  $|\Delta^{-1}(1) \cap R_2| = 2m - 1 + \beta + \lfloor \frac{2m-2}{3} \rfloor + \delta$ .

Since  $\delta = 0$ , we have

$$m \geq 3.$$

Since  $\delta = \gamma = \beta = 0$  and  $|\Delta^{-1}(1) \cap R_2| = 2m - 1 + \beta + \lfloor \frac{2m-2}{3} \rfloor + \delta$ , it follows that there are exactly  $2m - 1 + \lfloor \frac{2m-2}{3} \rfloor$  integers colored by 1 in  $R_2$ , all of them consecutive, with the remaining  $m - 1$  integers colored by 0. Thus

$$(20) \quad \Delta R_2 = 0^\lambda 1^{2m-1+\lfloor \frac{2m-2}{3} \rfloor} 0^{m-1-\lambda},$$

for some  $\lambda \in [0, m - 1]$ . Since  $\alpha = \lfloor \frac{2m+1}{3} \rfloor > 0 = \beta$ , we cannot have (iii) holding in Lemma 2.1. Since  $\alpha > 0$  and  $\beta = 0$ , we cannot have (ii) holding in Lemma 2.1. Thus Lemma 2.1(i) must hold.

If  $\beta = m - 1 - \alpha$ , then  $\alpha = m - 1$  in view of  $\beta = 0$ . Moreover, the string  $H_0 0 H_1$  contains exactly  $m - 1 - \alpha = \beta = 0$   $c_1$ 's. Thus  $\Delta R_1 = c_1^{m-1-\nu} c_0^{2m-2} c_1^{1+\nu}$ . But now, in view of  $m \geq 3$  and (20), it

is clear that  $[m - \nu, 2m - \nu - 1]$ ,  $[\text{first}(1, R_2), \text{first}(1, R_2) + m - 1]$  and  $[\text{first}(1, R_2) + m, \text{first}(1, R_2) + 2m - 1]$  are three monochromatic sets each of diameter  $m - 1$ , yielding a monochromatic solution to  $p(m, m, m; 2)$ . Therefore we may instead assume  $0 = \beta < m - 1 - \alpha$ , implying

$$(21) \quad \mu = \nu = \beta = 0, \quad \alpha = \lfloor \frac{2m+1}{3} \rfloor \leq m-2 \quad \text{and} \quad m \geq 5,$$

where the final inequality follows from the second. In this case, Lemma 2.1 implies

$$(22) \quad \Delta R_1 = c_1^{m-1} H_0 c_0^{m-1} c_1.$$

We divide the remainder of the proof of Step 1 into two cases.

**Case 1.1:**  $c_0 = 1$ .

In this case, we see from (20) and (22) that

$$(23) \quad \Delta[2m-1, 6m-4 - \lfloor \frac{2m-2}{3} \rfloor] = 1^{m-1} 0^{\lambda+1} 1^{2m-2+\alpha} 0^{m-1-\lambda}.$$

Recall from (21) that

$$3 \leq \alpha = \lfloor \frac{2m+1}{3} \rfloor \leq m-2$$

and recall from Lemma 2.2(ii)(c) that  $A_3 \subseteq [1, m + \alpha + \beta] \subseteq [1, 2m - 2]$ , where the second inclusion follows from (21), with  $\text{diam } A_3 \leq m + \alpha + \beta - 1 = m + \alpha - 1$ .

Observe that  $m + \lambda + 2m - 2 + \alpha \geq 3m - 2 + \alpha \geq 2m + 2\alpha$  in view of  $\alpha \leq m - 2$ . Consequently, if  $\lambda + 1 \leq \alpha$ , then it follows from (23) that  $C = [2m - 1, 3m - 3] \cup \{3m - 2 + \alpha\}$  and  $D = [3m - 1 + \alpha, 4m - 3 + \alpha] \cup \{4m - 2 + 2\alpha\}$  will be monochromatic  $m$ -subsets with  $\text{diam } C = \text{diam } D = m + \alpha - 1$ , in which case  $A_3$ ,  $C$  and  $D$  form a monochromatic solution to  $p(m, m, m; 2)$ . Therefore we may instead assume

$$(24) \quad \lambda + 1 \geq \alpha + 1.$$

Observe that  $2m - 3 + \alpha \geq 2m \geq m + \lambda + 1$  in view of  $\alpha \geq 3$  and  $\lambda \leq m - 1$ . Consequently, it follows from (23) that  $C = [2m - 1, 3m - 3] \cup \{3m + \lambda - 1\}$  and  $D = [3m + \lambda, 4m + \lambda - 2] \cup \{4m + 2\lambda\}$  are monochromatic  $m$ -subsets with  $\text{diam } C = \text{diam } D = m + \lambda \geq m + \alpha$  (in view of (24)), in which case  $A_3$ ,  $C$  and  $D$  form a monochromatic solution to  $p(m, m, m; 2)$ , completing the case.

**Case 1.2:**  $c_0 = 0$

In this case, we see from (20) and (22) that

$$(25) \quad \Delta[2m-1, 6m-4 - \lfloor \frac{2m-2}{3} \rfloor] = 0^{m-1} 1 0^\lambda 1^{2m-2+\alpha} 0^{m-1-\lambda}.$$

Recall from Lemma 2.2(ii)(c) that  $A_3 \subseteq [1, m + \alpha + \beta] \subseteq [1, 2m - 2]$  is a monochromatic  $m$ -subset, where the second inclusion follows from (21), with  $\text{diam } A_3 \leq m + \alpha - 1$  (in view of  $\beta = 0$ ).

If  $\lambda \geq \alpha$ , then recall that  $\alpha \geq 1$  and observe that  $2m - 2 + \alpha \geq m + \alpha$ . As a result, we see from (25) that  $C = [2m - 1, 3m - 3] \cup \{3m - 2 + \alpha\}$  and  $D = [3m - 1 + \lambda, 4m - 3 + \lambda] \cup \{4m - 2 + \lambda + \alpha\}$  are monochromatic  $m$ -subsets with  $\text{diam } C = \text{diam } D = m + \alpha - 1$ , in which case  $A_3$ ,  $C$ , and  $D$  form a monochromatic solution to  $p(m, m, m; 2)$ .

If  $m - \alpha - 1 \leq \lambda \leq \alpha - 1$ , then  $m - 1 + \alpha \geq m + \lambda$ . As a result, we see from (25) that  $C = \{3m - 2\} \cup [3m - 1 + \lambda, 4m - 3 + \lambda]$  and  $D = [4m - 2 + \lambda, 5m - 4 + \lambda] \cup \{5m - 3 + 2\lambda\}$  are monochromatic  $m$ -subsets with  $\text{diam } C = \text{diam } D = m + \lambda - 1 \geq 2m - 2 - \alpha$ , in which case  $A_1 \subseteq [1, 3m - 2 - \alpha] \subseteq [1, 3m - 3]$ ,  $C$  and  $D$  form a monochromatic solution to  $p(m, m, m; 2)$ .

Finally, if  $\lambda \leq m - \alpha - 1$ , then it follows from (25) that  $C = \{3m - 2\} \cup [4m - 2 - \alpha, 5m - 4 - \alpha]$  is a monochromatic  $m$ -subset with  $\text{diam } C = 2m - 2 - \alpha$ . Moreover, there are

$$2m - 2 + \alpha - (2m - 1 - \alpha - 1 - \lambda) = 2\alpha + \lambda \geq 2\alpha \geq 2m - 1 - \alpha$$

integers greater than  $5m - 4 - \alpha$  that are colored by 1, where the final inequality follows in view of  $3\alpha = 3\lfloor \frac{2m+1}{3} \rfloor \geq 3\frac{2m-1}{3} = 2m - 1$ . Thus, in view of (25), we conclude that  $D = [5m - 3 - \alpha, 6m - 5 - \alpha] \cup \{\text{last}(1, R_2)\}$  is a monochromatic  $m$ -subset with  $\text{diam } D \geq 2m - 2 - \alpha$ , in which case  $A_1 \subseteq [1, 3m - 2 - \alpha] \subseteq [1, 3m - 3]$ ,  $C$  and  $D$  form a monochromatic solution to  $p(m, m, m; 2)$ , completing the case and Step 1.

We may w.l.o.g. assume  $\Delta(3m - 1 - \beta) = 1$ . Then, since  $|\Delta^{-1}(1) \cap R_1| \geq m$  by Step 1, we must have  $\Delta([5m - 3 - \beta + \alpha, 6m - 4 + \lfloor \frac{2m-2}{3} \rfloor + \delta]) = \{0\}$  else, letting  $C = \text{first}_1^{m-1}(1, R_2) \cup \{\text{last}(1, R_2)\}$ , it follows that  $A_0, B_1$  and  $C$  form a monochromatic solution to  $p(m, m, m; 2)$ . But now we likewise have  $\Delta([3m - 1 - \beta, 4m - 2 - \alpha + \lfloor \frac{2m-2}{3} \rfloor + \delta]) = \{1\}$  else, letting  $C = \text{first}_1^{m-1}(0, R_2) \cup \{\text{last}(0, R_2)\}$ , it follows that  $A_0, B_1$  and  $C$  form a monochromatic solution to  $p(m, m, m; 2)$ . In summary,

$$(26) \quad \Delta\left([3m - 1 - \beta, 4m - 2 - \alpha + \lfloor \frac{2m-2}{3} \rfloor + \delta]\right) = \{1\} \quad \text{and}$$

$$(27) \quad \Delta\left([5m - 3 - \beta + \alpha, 6m - 4 + \lfloor \frac{2m-2}{3} \rfloor + \delta]\right) = \{0\}.$$

Note that both of these intervals contain  $m - \alpha + \beta + \lfloor \frac{2m-2}{3} \rfloor + \delta$  integers.

Let

$$\begin{aligned} C_1 &= [3m - 1 - \beta, 4m - 3 - \beta] \cup \{4m - 2 - \alpha + \lfloor \frac{2m-2}{3} \rfloor + \delta\} \quad \text{and} \\ C_2 &= [5m - 3 - \beta + \alpha, 6m - 5 - \beta + \alpha] \cup \{6m - 4 + \lfloor \frac{2m-2}{3} \rfloor + \delta\}. \end{aligned}$$

When  $\beta - \alpha + \lfloor \frac{2m-2}{3} \rfloor + \delta \geq 0$ , these are both monochromatic  $m$ -subsets of diameter  $\text{diam } C_1 = \text{diam } C_2 = m - 1 + \beta - \alpha + \lfloor \frac{2m-2}{3} \rfloor + \delta$ .

If  $\beta - \alpha + \lfloor \frac{2m-2}{3} \rfloor + \delta \geq m - 1 - \alpha$ , then  $C_1$  and  $C_2$  are monochromatic  $m$ -subsets with  $\text{diam } C_1 = \text{diam } C_2 \geq 2m - 2 - \alpha$ , in which case  $A_1, C_1$  and  $C_2$  form a monochromatic solution to  $p(m, m, m; 2)$ . If  $\beta - \alpha + \lfloor \frac{2m-2}{3} \rfloor + \delta \geq \frac{m-2+\beta}{2}$ , then  $C_1$  and  $C_2$  are monochromatic  $m$ -subsets with  $\text{diam } C_1 = \text{diam } C_2 \geq m + \lceil \frac{m-2+\beta}{2} \rceil - 1$ , in which case  $A_2, C_1$  and  $C_2$  form a monochromatic solution to  $p(m, m, m; 2)$ . In summary, we may instead assume the contrary of

both these inequalities, in turn yielding

$$(28) \quad \beta \leq \frac{m-2}{3} - \delta < m-1 \quad \text{and}$$

$$(29) \quad \alpha \geq \frac{m+1}{6} + \frac{\beta}{2} + \delta > 0.$$

From (29) and (28), we derive that

$$(30) \quad \alpha > \frac{m-2}{6} + \frac{\beta}{2} = \frac{1}{2} \left( \frac{m-2}{3} + \beta \right) \geq \beta.$$

Thus Lemma 2.1(iii) cannot hold.

If  $\text{first}(0, R_2) > 5m - 2 - \beta$ , then  $\Delta([3m - 1 - \beta, 5m - 2 - \beta]) = \{1\}$ , in which case  $D_1 = [3m - 1 - \beta, 4m - 2 - \beta]$ ,  $D_2 = [4m - 1 - \beta, 5m - 2 - \beta]$  and  $D_3 = \text{first}_1^m(0, R_2)$  form a monochromatic solution to  $p(m, m, m; 2)$  in view of Step 1. Therefore we may instead assume

$$(31) \quad \text{first}(0, R_2) \leq 5m - 2 - \beta.$$

**Step 2:** Lemma 2.1(i) holds.

Since Lemma 2.1(iii) does not hold as noted above, assume to the contrary that Lemma 2.1(ii) holds instead. We divide the step into two cases.

**Case 2.1:**  $c_1 = 1$ .

In this case, Lemma 2.1(ii) and (26) yield

$$\Delta\left([2m - 1, 4m - 2 - \alpha + \lfloor \frac{2m-2}{3} \rfloor + \delta]\right) = \{1\}.$$

Recall that  $\alpha + \beta \leq m - 1$ . Hence, letting  $C = [3m - 1 - \alpha - \beta, 4m - 3 - \alpha - \beta] \cup \{4m - 2 - \alpha\}$ , it follows that  $C$  is a monochromatic  $m$ -set with  $\text{diam}(C) = m - 1 + \beta$ . Consequently, we must have  $\text{first}(0, R_2) \geq 5m - 2 - \beta + \lfloor \frac{2m-2}{3} \rfloor + \delta$ , for otherwise, letting  $D = \text{first}_1^{m-1}(0, R_2) \cup \{6m - 4 + \lfloor \frac{2m-2}{3} \rfloor + \delta\}$ , it follows in view of Step 1 and Lemma 2.2(ii)(b) that  $A_1 \subseteq [1, 3m - 2 - \alpha - \beta]$ ,  $C$  and  $D$  form a monochromatic solution to  $p(m, m, m; 2)$ . However, since  $\lfloor \frac{2m-2}{3} \rfloor + \delta \geq 1$ , this is contrary to (31), completing the case.

**Case 2.2:**  $c_1 = 0$ .

In this case, we have  $\beta \geq 1$  (in view of (29) and Lemma 2.1(ii)) and

$$\Delta R_1 = 1^{m-\alpha-\beta-1} 0 H_2 0^{m-\beta}$$

with  $\Delta[m - \alpha - \beta + 1, 2m - 2] = H_2$  a string of length  $m - 2 + \beta + \alpha$  that contains exactly  $\beta - 1$  0's and  $m - 1 + \alpha$  1's (in view of  $\alpha > 0$  from (29)). In particular,  $\text{last}_{m-1}(1, R_1) \geq m - \beta + 1$ .

Let  $y \leq m - \beta + 1$  be the largest integer with  $\Delta(y) = 1$ . Since there are at most  $\beta - 1$  integers colored by 0 in  $[m - \alpha - \beta + 1, m - \beta + 1] \subseteq [m - \alpha - \beta + 1, 2m - 2]$  (with the inclusion in view of  $\beta \geq 1$ ), which is an interval of length  $\alpha + 1 \geq \beta + 2$  (in view of (30)), it follows that

$$(32) \quad m - 2\beta + 2 \leq y \leq m - \beta + 1.$$

Let  $B = \{y\} \cup \text{last}_1^{m-2}(1, R_1) \cup \{3m - 1 - \beta\}$ . Since  $\text{last}_{m-1}(1, R_1) \geq m - \beta + 1$ , it follows in view of (32) that  $B$  is a monochromatic  $m$ -subset with

$$2m - 2 \leq \text{diam}(B) \leq 2m - 3 + \beta.$$

If  $\text{first}(0, R_2) \leq 4m - 1 + \lfloor \frac{2m-2}{3} \rfloor - \beta + \delta$ , then  $A_0, B$  and  $\text{first}_1^{m-1}(0, R_2) \cup \{\text{last}(0, R_2)\}$  form a monochromatic solution to  $p(m, m, m; 2)$  in view of Step 1 and (27). Therefore we may instead assume

$$\text{first}(0, R_2) \geq 4m + \lfloor \frac{2m-2}{3} \rfloor - \beta + \delta > 4m - 2,$$

with the latter inequality in view of (28). Hence, letting  $C = [3m - 1 - \beta, 4m - 3 - \beta] \cup \{4m - 2\}$ , it follows that  $C$  is a monochromatic  $m$ -subset with  $\text{diam } C = m - 1 + \beta$ . On the other hand, in view of Step 1, (31) and (27), we have  $D = \text{first}_1^{m-1}(0, R_2) \cup \{\text{last}(0, R_2)\}$  being a monochromatic  $m$ -subset with

$$\text{diam } D \geq m - 2 + \beta + \lfloor \frac{2m-2}{3} \rfloor + \delta \geq m + \beta - 1.$$

Thus  $A_1, C$  and  $D$  form a monochromatic solution to  $p(m, m, m; 2)$  in view of Lemma 2.2(ii)(b) and (29), completing Step 2.

Recall, in view of (26) and (27), that  $C_1$  and  $C_2$  are both monochromatic  $m$ -subsets of diameter  $\text{diam } C_1 = \text{diam } C_2 = m - 1 + \beta - \alpha + \lfloor \frac{2m-2}{3} \rfloor + \delta$  when  $\beta - \alpha + \lfloor \frac{2m-2}{3} \rfloor + \delta \geq 0$ .

If  $\beta - \alpha + \lfloor \frac{2m-2}{3} \rfloor + \delta \geq m - 1 - \alpha - \mu$ , then  $C_1$  and  $C_2$  are monochromatic  $m$ -subsets with  $\text{diam } C_1 = \text{diam } C_2 \geq 2m - 2 - \alpha - \mu$ , in which case  $A_1, C_1$  and  $C_2$  form a monochromatic solution to  $p(m, m, m; 2)$  in view of Lemma 2.2(ii)(c) and Step 2. Likewise, if  $\beta - \alpha + \lfloor \frac{2m-2}{3} \rfloor + \delta \geq \alpha + \beta$ , then  $C_1$  and  $C_2$  are monochromatic  $m$ -subsets with  $\text{diam } C_1 = \text{diam } C_2 \geq m + \alpha + \beta - 1$ , in which case  $A_3, C_1$  and  $C_2$  form a monochromatic solution to  $p(m, m, m; 2)$ . In summary, we may instead assume the contrary of both these inequalities, in turn yielding

$$(33) \quad \beta \leq \frac{m-2}{3} - \mu - \delta \quad \text{and}$$

$$(34) \quad \alpha \geq \frac{m+\delta}{3} > 0.$$

In particular, (33) and  $\beta \geq 0$  yield

$$m \geq 3.$$

**Step 3:**  $c_1 = 1$ .

Assume by contradiction that  $c_1 = 0$ . We divide the step into two cases.

**Case 3.1:**  $\beta = m - 1 - \alpha$ .

In this case, Lemma 2.1(i) yields

$$\Delta[1, 3m - 2 - \beta] = 0^{m-1-\beta-\nu} H_0 1 H_1 0^{1+\nu}$$

with the string  $\Delta[m - \beta - \nu, 2m - 2 - \nu - \mu] = H_0$  containing exactly  $\beta - \mu = m - 1 - \alpha - \mu$  0's. We trivially have  $m - 1 - \beta - \nu \geq 0$  and  $m - 1 - \alpha - \mu = \beta - \mu \geq 0$  as these quantities from

Lemma 2.1(i) must be nonzero, implying  $\nu + \mu \leq m - 1$ . Thus  $2m - 2 - \nu - \mu \geq m - 1$ , which means the string  $H_01$  covers the interval  $[m - \beta - \nu, m]$ , thus ensuring there are at most  $\beta - \mu$  integers colored by 0 in  $[m - \beta - \nu, m]$ . Since this interval contains at least  $\beta + 1$  elements, this ensures that there is some

$$(35) \quad y \in [m - \beta + \mu, m] \quad \text{with} \quad \Delta(y) = 1 \quad \text{and} \quad y \leq \text{last}_{m-1-\beta}(1, R_1),$$

where the later inequality follows by recalling from Lemma 2.1(i) that the string  $H_1$  contains exactly  $m - 2 - \beta$  1's. From (26) and  $\alpha = m - 1 - \beta$ , we know

$$\Delta([3m - 1 - \beta, 3m - 1 + \beta + \lfloor \frac{2m-2}{3} \rfloor + \delta]) = \{1\}.$$

Thus  $B = \{y\} \cup \text{last}_1^{m-2-\beta}(1, R_1) \cup [3m - 1 - \beta, 3m - 1]$  is a monochromatic  $m$ -subset (in view of (35)) with

$$2m - 1 \leq \text{diam } B \leq 2m - 1 + \beta.$$

If  $\text{first}(0, R_2) \leq 4m - 3 - \beta + \lfloor \frac{2m-2}{3} \rfloor + \delta$ , then  $A_0$ ,  $B$  and  $\text{first}_1^{m-1} \cup \{\text{last}(0, R_2)\}$  form a monochromatic solution to  $p(m, m, m; 2)$  in view of Step 1 and (27). Therefore we may instead assume

$$\text{first}(0, R_2) \geq 4m - 2 - \beta + \lfloor \frac{2m-2}{3} \rfloor + \delta > 4m - 2,$$

with the latter inequality following from (33). Thus  $C = [3m - 1 - \beta, 4m - 3 - \beta] \cup \{4m - 2\}$  is a monochromatic  $m$ -subset with  $\text{diam } C = m - 1 + \beta$ . On the other hand, in view of Step 1, (27) and (31), we see that  $D = \text{first}_1^{m-1}(0, R_2) \cup \{\text{last}(0, R_2)\}$  is a monochromatic  $m$ -subset with  $\text{diam } D \geq m - 2 + \beta + \lfloor \frac{2m-2}{3} \rfloor + \delta \geq m - 1 + \beta$ , whence  $A_1$ ,  $C$  and  $D$  give a monochromatic solution to  $p(m, m, m; 2)$  in view of Lemma 2.2(ii) and  $\alpha = m - 1 - \beta$ , completing the case.

**Case 3.2:**  $\beta < m - 1 - \alpha$ .

In this case,

$$(36) \quad \alpha \leq m - 2 - \beta \leq m - 2,$$

while Lemma 2.1(i) yields  $\mu = \nu = 0$  and

$$(37) \quad \Delta[1, 3m - 2 - \beta] = 0^{m-1-\beta} H_0 1^{m-1-\beta} 0$$

with the string  $\Delta[m - \beta, 2m - 2] = H_0$  containing exactly  $m - 1 - \alpha$  0's and exactly  $\alpha + \beta$  1's. Now  $m - 1 - \beta \geq \alpha + 1$  by case hypothesis, and  $4m - 2 - \alpha + \lfloor \frac{2m-2}{3} \rfloor \geq 5m - 3 - 2\alpha$  in view of (34). Thus, if  $\alpha \geq 2$ , then (37), (36) and (26) imply  $B = [3m - 1 - \alpha - \beta, 3m - 3 - \beta] \cup [4m - 3 - \alpha - \beta, 5m - 3 - 2\alpha - \beta]$  is a monochromatic  $m$ -subset with  $\text{diam } B = 2m - 2 - \alpha$  and  $B \subseteq [3m - 1 - \alpha - \beta, \text{first}(0, R_2) - 1]$ . On the other hand, if  $\alpha \leq 1$ , then (34) ensures that  $\alpha = 1$ ,  $m = 3$  (recall that we now know  $m \geq 3$ ),  $\beta = 0$  (in view of the case hypothesis  $\beta \leq m - 2 - \alpha$ ), and  $\text{diam } A_3 = m + \alpha + \beta - 1 = 2m - 2 - \alpha$  (by Lemma 2.2(ii)). In this case,  $B = \{3m - 3 - \beta\} \cup [4m - 3 - \alpha - \beta, 5m - 5 - \alpha - \beta]$  is a monochromatic  $m$ -subset with  $\text{diam } B = 2m - 2 - \alpha$  and  $\min B \geq m + \alpha + \beta + 1$ .



In view of Step 1, (27) and (31), it follows that  $C = \text{first}_1^{m-1}(0, R_2) \cup \{\text{last}(0, R_2)\}$  is a monochromatic  $m$ -subset with

$$(38) \quad \text{diam } C \geq m - 2 + \beta + \lfloor \frac{2m-2}{3} \rfloor + \delta \geq m - 2 + \lfloor \frac{2m-2}{3} \rfloor \geq 2m - 3 - \alpha,$$

with the latter inequality once more in view of (34). Thus  $A_1$  (if  $\alpha \geq 2$ ) or  $A_3$  (if  $\alpha = 1$ ),  $B$  and  $C$  will form a monochromatic solution to  $p(m, m, m; 2)$  unless equality holds in all the estimates used to derive (38). In particular, we must have  $\delta = 0$ ,  $\beta = 0$ ,  $\alpha = \lfloor \frac{m+1}{3} \rfloor$  and  $\text{first}(0, R_2) = 5m - 2 - \beta = 5m - 2$ .

Since  $\beta = 0$ , (37) gives

$$\Delta R_1 = 0^{m-1} H_0 1^{m-1} 0.$$

In view of  $\text{first}(0, R_2) = 5m - 2$  and  $\beta = 0$ , we have

$$\Delta([3m - 1, 5m - 3]) = \{1\}.$$

Thus, since  $\alpha \leq \frac{m+1}{3}$  and  $m \geq 3$  imply  $4m - 2 + 2\alpha \leq 4m - 2 + \frac{2m+2}{3} < 5m - 2$ , it follows that  $D_1 = [2m - 1, 3m - 3] \cup \{3m - 2 + \alpha\}$  and  $D_2 = [3m - 1 + \alpha, 4m - 3 + \alpha] \cup \{4m - 2 + 2\alpha\}$  are monochromatic  $m$ -subsets with  $\text{diam } D_1 = \text{diam } D_2 = m - 1 + \alpha$ . So  $A_3 \subseteq [1, m + \alpha + \beta] \subseteq [1, 2m - 2]$  (the inclusion follows from the case hypothesis),  $D_1$  and  $D_2$  form a monochromatic solution to  $p(m, m, m; 2)$  in view of Lemma 2.2(ii)(c) and  $\beta = 0$ , completing the case and Step 3.

In view of Steps 2 and 3 and Lemma 2.1(i), we now have

$$(39) \quad \Delta[1, 3m - 2 - \beta] = 1^{m-1-\beta-\nu} H_0 0 H_1 1^{1+\nu},$$

where  $H_0 = \Delta[m - \beta - \nu, 2m - 2 - \mu - \nu]$  is a string containing exactly  $m - 1 - \alpha - \mu$  1's and  $\alpha + \beta$  0's and  $H_1 = \Delta[2m - \mu - \nu, 3m - 3 - \beta - \nu]$  is a string containing exactly  $\mu$  1's and  $m - 2 - \beta$  0's. Let

$$R'_2 = [3m - 2 - \beta - \nu, 6m - 4 + \lfloor \frac{2m-2}{3} \rfloor]$$

and observe that  $\Delta([3m - 2 - \beta - \nu, 4m - 2 - \alpha + \lfloor \frac{2m-2}{3} \rfloor + \delta]) = \{1\}$  in view of (26) and (39).

If  $\text{first}(0, R_2) > 5m - 3 - \beta - \nu$ , then  $\Delta([3m - 2 - \beta - \nu, 5m - 3 - \beta - \nu]) = \{1\}$ , in which case  $D_1 = [3m - 2 - \beta - \nu, 4m - 3 - \beta - \nu]$ ,  $D_2 = [4m - 2 - \beta - \nu, 5m - 3 - \beta - \nu]$  and  $D_3 = \text{first}_1^m(0, R_2)$  form a monochromatic solution to  $p(m, m, m; 2)$  in view of Step 1. Therefore we may instead assume

$$(40) \quad \text{first}(0, R_2) \leq 5m - 3 - \beta - \nu.$$

Let

$$\gamma_0 = |\Delta^{-1}(0) \cap [\text{first}(1, R_2), \text{last}(1, R_2)]|.$$

We must have

$$(41) \quad \gamma_0 \leq |\Delta^{-1}(0) \cap R_2| + \alpha - m - \lfloor \frac{2m-2}{3} \rfloor - \beta \leq |\Delta^{-1}(0) \cap R_2| + \alpha - \frac{5m-4}{3} - \beta,$$

for otherwise  $C = \text{first}_1^{m-1}(1, R_2) \cup \{\text{last}(1, R_2)\}$  will be a monochromatic  $m$ -subset (in view of Step 1) with

$$\begin{aligned} \text{diam}(C) &= |\Delta^{-1}(1) \cap R_2| + \gamma_0 - 1 = (3m - 2 + \beta + \lfloor \frac{2m-2}{3} \rfloor + \delta - |\Delta^{-1}(0) \cap R_2|) + \gamma_0 - 1 \\ &\geq 2m - 2 + \alpha, \end{aligned}$$

in which case  $A_0$ ,  $B_1$  and  $C$  form a monochromatic solution to  $p(m, m, m; 2)$ .

**Step 4:**  $\nu \leq \alpha - 1$  and  $\alpha \geq \frac{2m+2}{3} + \mu + \nu + \delta$ .

To simplify notation, we proceed in two cases.

**Case 4.1:**  $\beta = m - \alpha + 1$ .

In this case, Lemma 2.2(ii)(c) shows  $\text{diam } A_1 \leq m + \beta - \mu - 1$  with  $A_1 \subseteq [1, 3m - 2 - \alpha - \beta]$ . Suppose  $\nu \geq \alpha - 1 - \lfloor \frac{2m-2}{3} \rfloor - \mu - \delta$ . Then (39) and (26) imply that the interval

$$I = [3m - 1 - \alpha - \beta + \mu + \lfloor \frac{2m-2}{3} \rfloor + \delta, 4m - 2 - \alpha + \lfloor \frac{2m-2}{3} \rfloor + \delta]$$

is entirely colored by 1. Since  $|I| \geq m + \beta - \mu$ , it follows that  $B = \text{first}_1^{m-1}(1, I) \cup \{\text{first}_{m+\beta-\mu}(1, I)\}$  is a monochromatic  $m$ -subset with  $\text{diam } B = m + \beta - \mu - 1$  and  $\min B \geq 3m - 1 - \alpha - \beta$ . Thus, in view of (40), (27) and Step 1, we see that  $A_1$ ,  $B$  and  $\text{first}^{m-1}(0, R_2) \cup \{\text{last}(0, R_2)\}$  form a monochromatic solution to  $p(m, m, m; 2)$ . So we may instead assume

$$\nu \leq \alpha - 2 - \lfloor \frac{2m-2}{3} \rfloor - \mu - \delta \leq \alpha - 2,$$

which rearranges to yield the desired bound for  $\alpha$ .

**Case 4.2:**  $\beta < m - \alpha + 1$ .

In this case, Lemma 2.1(i) implies that  $\mu = \nu = 0$ , so that  $\nu \leq \alpha - 1$  follows by (34). In particular,  $\min R'_2 \geq 3m - 1 - \alpha - \beta$ . Assume by contradiction that

$$(42) \quad \alpha \leq \frac{2m+1}{3} + \delta.$$

Then  $4m - 3 \leq 4m - 2 - \alpha + \lfloor \frac{2m-2}{3} \rfloor + \delta$ , so that (39) and (26) ensure

$$(43) \quad \Delta([3m - 2 - \beta, 4m - 3]) = \{1\},$$

which is an interval of length  $m + \beta \geq m$ . Let  $y = \text{last}_2(1, R_1)$ . Then  $y \leq 2m - 2$  by (39). If  $y < 2m - 2 - \beta$ , then  $D_1 = [2m - 2 - \beta, 3m - 3 - \beta]$  is a monochromatic  $m$ -subset (by (39)) with  $\text{diam } D_1 = m - 1$ , in which case  $D_1$ ,  $\text{first}_1^m(1, R'_2)$  and  $\text{first}_1^m(0, R'_2)$  form a monochromatic solution to  $p(m, m, m; 2)$  in view of Step 1 and (43). Therefore, we instead have  $2m - 2 - \beta \leq y \leq 2m - 2$ , which means  $D_2 = \{y\} \cup [3m - 1, 4m - 3]$  is a monochromatic  $m$ -subset (by (43)) with

$$2m - 1 \leq \text{diam } D_2 \leq 2m - 1 + \beta.$$

Thus, in view of Step 1 and (27), we must have

$$\text{first}(0, R_2) \geq 4m - 2 - \beta + \lfloor \frac{2m-2}{3} \rfloor + \delta,$$

else  $A_0$ ,  $D_2$  and  $\text{first}_1^{m-1}(0, R_2) \cup \{\text{last}(0, R_2)\}$  form a monochromatic solution to  $p(m, m, m; 2)$ . Consequently, since  $5m - 4 - \beta - \alpha \leq 4m - 3 - \beta + \lfloor \frac{2m-2}{3} \rfloor + \delta$  (by (34)), we see that  $D_3 = [3m-2-\beta, 4m-4-\beta] \cup \{5m-4-\beta-\alpha\}$  is a monochromatic  $m$ -subset with  $\text{diam } D_3 = 2m-2-\alpha$ . Moreover, in view of Step 1, (40) and (27), we see that  $D_4 = \text{first}_1^{m-1}(0, R_2) \cup \{\text{last}(0, R_2)\}$  is a monochromatic  $m$ -subset with

$$\text{diam } D_4 \geq m - 1 + \beta + \frac{2m-4}{3} + \delta \geq m - 1 + \frac{2m-4}{3} > 2m - 3 - \alpha,$$

where the latter inequality follows in view of (34). Thus  $A_1$ ,  $D_3$  and  $D_4$  form a monochromatic solution to  $p(m, m, m; 2)$  (in view of Lemma 2.2(ii) and  $\alpha \geq 1$ ), completing the case and Step 4.

In view of Step 4 and the basic inequality  $\alpha + \beta \leq m - 1$ , we have

$$m \geq 6 \quad \text{and} \quad \delta = 0,$$

$$R'_2 \subseteq [3m - 1 - \alpha - \beta, 6m - 4 + \lfloor \frac{2m-2}{3} \rfloor],$$

$$(44) \quad \alpha \geq \frac{2m+2}{3} + \mu + \nu \quad \text{and} \quad \beta \leq \frac{m-5}{3} - \mu - \nu.$$

**Step 5:**  $\text{last}(1, R_2) \geq 5m - 2 - \alpha - \beta$ .

Assume to the contrary that  $\text{last}(1, R_2) \leq 5m - 3 - \alpha - \beta$ . Then

$$I = [5m - 2 - \alpha - \beta, 6m - 4 + \lfloor \frac{2m-2}{3} \rfloor]$$

is an interval entirely colored by 0 with

$$(45) \quad |I| = m - 1 + \alpha + \beta + \lfloor \frac{2m-2}{3} \rfloor.$$

In particular,  $|I| \geq m - 1 + \frac{2m+2}{3} + \beta + \frac{2m-4}{3} \geq 2m + \beta + \frac{m-5}{3} \geq 2m + 2\beta$  by (44). Consequently, if  $\beta = m - 1 - \alpha$ , then  $A_1$ ,  $\text{first}_1^{m-1}(0, I) \cup \{\text{first}_{m+\beta}(0, I)\}$  and  $\text{first}_{m+\beta+1}^{2m+\beta-1}(0, I) \cup \{\text{first}_{2m+2\beta}(0, I)\}$  will form a monochromatic solution to  $p(m, m, m; 2)$  in view of Lemma 2.2(ii)(c). Therefore we may instead assume  $\beta < m - 1 - \alpha$ , in which case Lemma 2.1(i) implies  $\mu = \nu = 0$ .

In view of  $\beta < m - 1 - \alpha$ , we have  $\min I \geq 4m$ . Thus we must have

$$(46) \quad |\Delta^{-1}(1) \cap [3m - 2 - \beta, 4m - 3]| \leq m - 1,$$

for otherwise  $\text{first}_1^m(1, R'_2)$ ,  $\text{first}_1^{m-1}(0, I) \cup \{\text{first}_{m+\beta}(0, I)\}$  and  $\text{first}_{m+\beta+1}^{2m+\beta-1}(0, I) \cup \{\text{first}_{2m+2\beta}(0, I)\}$  will form a monochromatic solution to  $p(m, m, m; 2)$ . But this means

$$(47) \quad \text{first}_{\beta+1}(0, R_2) \leq 4m - 3.$$

By (39) and  $\mu = 0$ , we know  $\Delta[2m - 1, 3m - 3 - \beta] = 0^{m-1-\beta}$ . Thus

$$C = \text{first}_1^{m-1}(0, [2m - 1, 6m - 4 + \lfloor \frac{2m-2}{3} \rfloor]) \cup \{5m - 2 - \alpha - \beta\}$$

is a monochromatic  $m$ -subset (in view of (47)) with  $\text{diam } C = 3m - 3 - \alpha - \beta \geq 2m - 1$  (in view of  $\beta < m - 1 - \alpha$ ). On the other hand,

$$D = [5m - 1 - \alpha - \beta, 6m - 3 - \alpha - \beta] \cup \{6m - 4 + \lfloor \frac{2m-2}{3} \rfloor\}$$

is a monochromatic  $m$ -subset (in view of (45)) with  $\text{diam } D = m - 3 + \alpha + \beta + \lfloor \frac{2m-2}{3} \rfloor$ . However, in view of  $\beta \geq 0$  and (44), we have  $m - 3 + \alpha + \beta + \lfloor \frac{2m-2}{3} \rfloor \geq 3m - 3 - \alpha - \beta$ . Hence  $A_0$ ,  $C$  and  $D$  form a monochromatic solution to  $p(m, m, m; 2)$ , completing Step 5.

Let  $\beta' = m - 1 - \alpha \geq \beta$ . In view of (44), we have

$$(48) \quad \beta \leq \beta' \leq \frac{m-5}{3} - \mu - \nu.$$

Let  $y$  be the least integer such that  $\Delta(y) = 1$ ,  $y \geq \text{first}_m(1, R'_2)$  and

$$y \geq \text{first}(1, R'_2) + m + \beta' - \mu - 1 = 5m - 4 - \alpha - \beta - \mu - \nu.$$

Note  $y$  exists in view of Steps 1 and 5. Let

$$R_3 = [3m - 2 - \beta - \nu, y]$$

and let

$$\gamma'_0 = |\Delta^{-1}(0) \cap R_3| \quad \text{and} \quad \gamma''_0 = |\Delta^{-1}(0) \cap [\text{first}(1, R_3), \text{last}_2(1, R_3)]|.$$

Note that  $\text{first}(1, R_3) = \min R_3 = \min R'_2 = 3m - 2 - \beta - \nu$ .

If  $|\Delta^{-1}(1) \cap R_3| > m$ , then we must have  $|\Delta^{-1}(1) \cap R_3| - 1 + \gamma''_0 \leq m + \beta' - \mu - 1$ , else the minimality of  $y$  will be contradicted by  $\text{last}_2(1, R_3)$ . Thus

$$(49) \quad |\Delta^{-1}(1) \cap R_3| \leq m + \max\{0, \beta' - \mu - \gamma''_0\} \leq m + \beta' - \mu.$$

Consequently, if  $\text{last}(0, R_3) - \text{first}(0, R_3) \geq m - 2 + \max\{\gamma'_0, \beta' - \mu\}$ , then  $[\text{first}(0, R_3), \text{last}(0, R_3)]$  is an interval of length at least  $m - 1 + \gamma'_0$  that, by definition of  $\gamma'_0$ , can contain at most  $\gamma'_0$  integers colored by 0 and, consequently, must contain at least  $m - 1$  integers colored by 1. Since  $y > \text{last}(0, R_3)$  and is also colored by 1, this would mean there are no more than  $|\Delta^{-1}(1) \cap R_3| - m \leq \beta' - \mu \leq m - 1 - \alpha$  (with the second inequality by (49)) integers colored by 1 in  $[\min R_3, \text{first}(0, R_3) - 1] = [3m - 2 - \beta - \nu, \text{first}(0, R_3) - 1]$ . However, by (26) and (39), we know the first  $m - \alpha + \beta + \lfloor \frac{2m-2}{3} \rfloor + 1 + \nu \geq m - \alpha$  consecutive integers of  $R_3$  are colored by 1, making this impossible. Therefore we instead conclude that  $\text{last}(0, R_3) - \text{first}(0, R_3) \leq m - 3 + \max\{\gamma'_0, \beta' - \mu\}$ , in which case it is easily seen that there is a monochromatic  $m$ -subset  $C_0 \subseteq [3m - 2 - \beta - \nu, y] \subseteq [3m - 1 - \alpha - \beta, y]$  (where the second inclusion follows from Step 4) with

$$(50) \quad \text{diam } C_0 = m - 1 + \max\{\gamma'_0, \beta' - \mu\}$$

(since  $\text{diam } R_3 \geq m - 1 + \max\{\gamma'_0, \beta' - \mu\}$  in view of the definitions of  $R_3$ ,  $y$  and  $\gamma'_0$ ).

**Step 6:**  $|\Delta^{-1}(0) \cap R_2| \geq m + \beta + 1$ .

Suppose by contradiction that  $|\Delta^{-1}(0) \cap R_2| \leq m + \beta$ , so that

$$(51) \quad |\Delta^{-1}(1) \cap R_2'| \geq (3m - 1 + \beta + \frac{2m - 4}{3} + \nu) - m - \beta = \frac{8m - 7}{3} + \nu.$$

Then it follows from (49) and (51) that there are at least  $\frac{8m-7}{3} - m - \beta' + \mu + \nu = \frac{5m-7}{3} - \beta' + \mu + \nu$  integers colored by 1 that are greater than  $y$ . Thus we must have

$$(52) \quad \frac{5m - 7}{3} - \beta' + \mu + \nu \leq m - 1 + \max\{\gamma'_0, \beta' - \mu\},$$

for otherwise  $A_1, C_0$  and  $\text{first}_1^{m-1}(1, R_2 \setminus R_3) \cup \{\text{last}(1, R_2)\}$  will be a monochromatic solution to  $p(m, m, m; 2)$ .

If  $\gamma'_0 \leq \beta' - \mu$ , then (52) implies  $\beta' \geq \frac{m-2}{3} + \mu$ , contrary to (48). On the other hand, if  $\gamma'_0 \geq \beta' - \mu + 1$ , then (52) instead yields

$$\frac{2m - 4}{3} - \beta' + \mu + \nu \leq \gamma'_0 \leq \gamma_0 \leq \alpha - \frac{2m - 4}{3},$$

where the final inequality follows from (41) and the assumption  $|\Delta^{-1}(0) \cap R_2| \leq m + \beta$ . Rearranging the above inequality and applying the estimate (48), we find that  $\alpha \geq m - 1 + 2\mu + 2\nu$ . Since we trivially have  $\beta + \alpha \leq m - 1$ , we conclude that  $\alpha = m - 1$  and  $\beta = \mu = \nu = 0$  with equality holding in all estimates used to derive the bound  $\alpha \geq m - 1 + 2\mu + 2\nu$ . In particular,  $0 = m - 1 - \alpha = \beta' = \frac{m-5}{3} - \mu - \nu = \frac{m-5}{3}$ , contradicting that we now have  $m \geq 6$ . This completes Step 6.

**Step 7:**  $\gamma'_0 \geq \beta + 2$ .

Suppose by contradiction that  $\gamma'_0 \leq \beta + 1 \leq \beta' + 1$ . Then

$$m - 1 + \beta' - \mu \leq \text{diam } C_0 \leq m + \beta'$$

by (50). Also, in view of Step 6, there are at least  $m$  integers colored by 0 greater than  $y$ . As a result,

$$\text{first}(0, R_2 \setminus R_3) \geq 5m - 3 + \lfloor \frac{2m - 2}{3} \rfloor - \beta',$$

for otherwise  $A_1, C_0$  and  $\text{first}_1^{m-1}(0, R_2 \setminus R_3) \cup \{6m - 4 + \lfloor \frac{2m-2}{3} \rfloor\}$  form a monochromatic solution to  $p(m, m, m; 2)$  in view of (27) and Lemma 2.2(ii). Thus

$$(53) \quad \Delta([y, 5m - 4 + \lfloor \frac{2m - 2}{3} \rfloor - \beta']) = \{1\}.$$

We handle two cases.

**Case 7.1:**  $|\Delta^{-1}(1) \cap R_3| = m$ .

In this case, the assumption  $\gamma'_0 \leq \beta + 1$  yields

$$y = \min R_2 + |\Delta^{-1}(1) \cap R_3| + \gamma'_0 - 1 \leq 4m - 2 - \nu,$$

while (48) implies  $5m - 3 + \beta' \leq 5m - 4 + \lfloor \frac{2m-2}{3} \rfloor - \beta'$ . Thus

$$D = [4m - 2, 5m - 4] \cup \{5m - 3 + \beta'\}$$

is a monochromatic  $m$ -subset by (53) with  $\text{diam } D = m - 1 + \beta'$ . Consequently, we must now have

$$\text{first}(0, R_2 \setminus R_3) \geq 5m - 2 + \lfloor \frac{2m-2}{3} \rfloor - \beta',$$

for otherwise  $A_1, D$  and  $\text{first}_1^{m-1}(0, R_2 \setminus R_3) \cup \{6m - 4 + \lfloor \frac{2m-2}{3} \rfloor\}$  form a monochromatic solution to  $p(m, m, m; 2)$  in view of Step 6, (27), and Lemma 2.2(ii).

In consequence,

$$D' = [4m - 1, 5m - 3] \cup \{5m - 3 + \lfloor \frac{2m-2}{3} \rfloor - \beta'\}$$

is a monochromatic  $m$ -subset with  $\min D' > y$  and  $\text{diam } D' \geq m - 2 + \frac{2m-4}{3} - \beta' > m - 1 + \beta'$  in view of (48). Thus  $A_1, C_0$  and  $D'$  form a monochromatic solution to  $p(m, m, m; 2)$  in view of Lemma 2.2(ii), completing the case.

**Case 7.2:**  $|\Delta^{-1}(1) \cap R_3| > m$ .

In this case, (49) and the assumption  $\gamma'_0 \leq \beta + 1$  together imply that

$$\begin{aligned} y &= \min R_2 + |\Delta^{-1}(1) \cap R_3| + \gamma'_0 - 1 \\ &\leq 3m - 2 - \nu + |\Delta^{-1}(1) \cap R_3| \leq 4m - 2 + \beta' - \gamma''_0 - \mu - \nu \end{aligned}$$

and that  $B_0 = \text{first}_1^m(1, R_3)$  is a monochromatic  $m$ -subset with

$$\text{diam } B_0 \leq m + \gamma''_0 - 1 \quad \text{and} \quad \max B_0 < y.$$

Consequently, since  $5m - 3 + \beta' \leq 5m - 4 + \lfloor \frac{2m-2}{3} \rfloor - \beta'$  by (48), it follows from (53) that

$$C = [4m - 2 + \beta' - \gamma''_0, 5m - 4 + \beta' - \gamma''_0] \cup \{5m - 3 + \beta'\}$$

is a monochromatic  $m$ -subset with  $\min C \geq y > \max B_0$  and  $\text{diam } C = m + \gamma''_0 - 1$ . Thus we must have  $\text{first}(0, R_2 \setminus R_3) \geq 5m - 2 + \lfloor \frac{2m-2}{3} \rfloor - \gamma''_0$  else  $B_0, C$  and  $\text{first}_1^{m-1}(0, R_2 \setminus R_3) \cup \{6m - 4 + \lfloor \frac{2m-2}{3} \rfloor\}$  will be a monochromatic solution to  $p(m, m, m; 2)$ . But that means

$$D = [4m - 1 + \beta' - \gamma''_0, 5m - 3 + \beta - \gamma''_0] \cup \{5m - 3 + \lfloor \frac{2m-2}{3} \rfloor - \gamma''_0\}$$

is a monochromatic  $m$ -subset with  $\min D > y$  and  $\text{diam } D \geq m - 2 + \frac{2m-4}{3} - \beta' > m + \beta' - 1$  (in view of (48)). Hence  $A_1, C_0$  and  $D$  form a monochromatic solution to  $p(m, m, m; 2)$  in view of Lemma 2.2(ii), completing the case and Step 7.

Let  $z = \text{first}_{\beta+1}(0, R_2)$  and let  $z' = \text{first}_{\beta+2}(0, R_2)$ . In view of Step 6, there are at least  $m$  integers colored by 0 greater than  $z$ . In view of Step 7, we have  $z, z' \in R_3$  with  $z < z' < y$ . Thus, in view of the definition of  $z'$  and (49), we have

$$\begin{aligned} z' &\leq \min R_3 + |\Delta^{-1}(1) \cap R_3 \setminus \{y\}| + |\Delta^{-1}(0) \cap [\min R_3, z']| - 1 \\ &= \min R_3 + |\Delta^{-1}(1) \cap R_3 \setminus \{y\}| + |\Delta^{-1}(0) \cap [\min R_2, z']| - 1 \\ &= \min R_3 + |\Delta^{-1}(1) \cap R_3| + \beta \\ &\leq 4m - 2 - \nu + \beta' - \mu. \end{aligned}$$

Thus  $D = \{z'\} \cup \text{last}_1^{m-1}(0, R_2)$  is a monochromatic  $m$ -subset with  $\min D = z' > z$  and

$$(54) \quad \text{diam } D \geq 2m - 2 + \frac{2m - 4}{3} - \beta' + \mu + \nu \geq 2m - 2 + \frac{m + 1}{3} + 2\mu + 2\nu,$$

where the second inequality follows from (48).

Recall from (39) and the definition of  $z$  that  $\Delta(2m - 1 - \mu - \nu) = 0$  with the interval  $[2m - 1 - \mu - \nu, z]$  containing exactly  $m$  integers colored by 0 and at most

$$\mu + |\Delta^{-1}(1) \cap R_3 \setminus \{y\}| \leq \mu + m + \beta' - \mu - 1 = m + \beta' - 1$$

integers colored by 1 (in view of (49)). Thus, if there are at least  $m - 1$  integers colored by 1 in  $[2m - 1 - \mu - \nu, z]$ , then  $C = \text{first}_1^m(0, [2m - 1 - \mu - \nu, z])$  will be a monochromatic  $m$ -subset with  $2m - 2 \leq C \leq 2m - 2 + \beta'$ , in which case  $A_0, C$  and  $D$  will form a monochromatic solution to  $p(m, m, m; 2)$  in view of (54) and (48). Therefore we may instead assume there are at most  $m - 2$  integers colored by 1 in  $[2m - 1 - \mu - \nu, z]$ .

In view of (26) and (39), there are at least  $m - \alpha + \beta + \frac{2m-4}{3} + 1 + \nu \geq m - \alpha + 1$  integers colored by 1 in  $[2m - 1 - \mu - \nu, z]$ . Furthermore, in view of (39), we know that there are  $\alpha + \beta$  integers colored by 0 less than  $2m - 1 - \mu - \nu$  with at most  $m - 1 - \alpha$  integers colored by 1 between  $\text{first}(0, R_1)$  and  $2m - 1 - \mu - \nu$ . Consequently, there exists a monochromatic  $m$ -subset  $C' \subseteq [\text{first}(0, R_1), z]$  with  $2m - 2 \leq \text{diam } C' \leq 2m - 2 + (m - 1 - \alpha) = 2m - 2 + \beta'$ . Indeed, simply take  $C$  as defined in the previous paragraph and replace  $\min C$  with the maximal integer colored by 0 such that the resulting  $m$ -set  $C'$  has  $\text{diam } C' \geq 2m - 2$ . This integer exists as  $|\text{first}(0, R_1), z| \geq |\Delta^{-1}(0) \cap [1, z]| + |\Delta^{-1} \cap [\text{first}(0, R_1), z]| \geq m + \alpha + \beta + m - \alpha + 1 \geq 2m + \beta + 1$ . But now  $A_0, C'$  and  $D$  form a monochromatic solution to  $p(m, m, m; 2)$  in view of (54) and (48), completing the proof.  $\square$

### 3. ACKNOWLEDGEMENT

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